Lagrange Principle and the Distribution of Wealth

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Abstract

The Lagrange principle $L = f + \lambda\ g \rightarrow$ maximum! is used to maximize a function f(x) under a constraint g(x). Economists regard f(x) = U as a rational production function, which has to be maximized under the constraint of prices g(x). In physics f(x) = ln P is regarded as entropy of a stochastic system, which has to be maximized under constraint of energy g(x). In the discussion of wealth distribution it may be demonstrated that both aspects will apply. The stochastic aspect of physics leads to a Boltzmann distribution of wealth, which applies to the majority of the less affluent population. The rational approach of economics leads to a Pareto distribution, which applies to the minority of the super rich. The boundary corresponds to an economic phase transition similar to the liquid – gas transition in physical systems.

Introduction

Since the work of Pareto (1) as long ago as 1897, it has been known that economic distributions strictly follow power law decays. These distributions have been observed across a wide variety of economic and financial data. More recently, Roegen (2), Foley (3), Weidlich, (4) Mimkes (5,6), Levy and Solomon (7,8), Solomon and Richmond (9), Mimkes and Willis (10), Yakovenko (11), Clementi and Gallegatti (12) and Nirei and Souma (13) have proposed statistical models for economic distributions. In this paper the Lagrange principle is applied to recent data of wealth in different countries.

Distribution of property in Germany 1993 (DIW)
Property data for Germany (1993) have been published (14) by the German Institute of Economics (DIW). The data shows the number N(x) of households and the amount of capital K(x) in each property class (k), table 1.

Property distribution in Germany 1993 (DIW estimated)		
Total property or capital (K_0)	9920	Bill. DM
Number (N_0) of households	35,6	Mill.
Mean property (λ) per household	278	kDM / Hh

Property class (k) in kDM	Number N(x) of Hh in %	Property K(x) of Hh in %
k = 1: <100	46,0	9,5
k = 2: 100-250	24,7	17,6
k = 3: 250-500	20,3	28,1
k = 4: 500-1000	6,3	16,8
k = 5 >1000	2,7	28,0

Table. 1 *Distribution of property in different property classes (k) for households in Germany (1993), according to study by the DIW (14).*

The data are generally presented by a Lorenz distribution. Fig. 1 shows the distribution of capital K vs. the number N of households according to table 1.

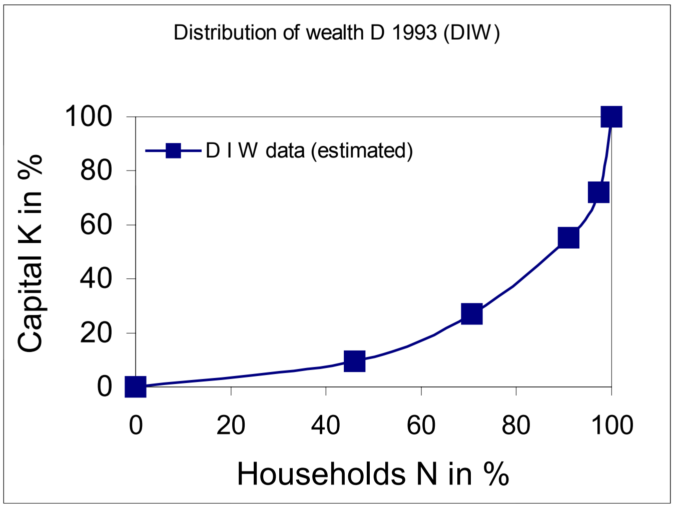

Fig. 1 *Lorenz distribution sum of capital K vs. sum of households N in Germany 1993, data from DIW (14), solid line according to Eq.(2.9).*

The distribution of the N(k) households and the amount of capital K(k) = k N(k) in each property class (k) are now discussed by the Lagrange principle for two models, an economic and a physical approach.

1.0 Lagrange principle

The Lagrange equation

$$\mathbf{f - \lambda\, g \qquad \rightarrow \qquad maximum!} \qquad \mathbf{(1.0)}$$

applies to all functions (f) that are to be maximized under constraints (g). The factor λ is called Lagrange parameter.

2.0 Calculation of the Boltzmann distribution

In the view of most physicists economic interactions are stochastic and the probability P is to be maximized under constraints of capital according to the Lagrange principle

$$\mathbf{\ln P(x) \; - \; \lambda \; \Sigma_j \, w_j \, x_j \rightarrow \quad maximum!} \qquad \mathbf{(2.0).}$$

ln P is the logarithm of probability $P(x_j)$ or entropy that will be maximized under the constraints of the total capital in income $\Sigma\, w_j\, x_j$. The variable (x_j) is the relative number of people in the income class (w_j). The Lagrange factor $\lambda = 1 / \langle w \rangle$ is equivalent to the mean income $\langle w \rangle$ per person. Distributing N households to (w_j) property classes is a question of combinatorial statistics,

$$\mathbf{P \quad = \quad N! \, / \; \Pi \, (N_j \, !)} \qquad \mathbf{(2.1).}$$

Using Sterling´s formula ($\ln N! = N \ln N - N$) and $x = N_j / N$ we may change Eq.(2.0) to

$$\mathbf{- \; \Sigma_j \;\; x_j \ln x_j - \lambda \;\; \Sigma_j \, w_j \, x_j \; \rightarrow \quad maximum!} \qquad \mathbf{(2.2).}$$

At equilibrium (maximum) the derivative of equation (2.2) with respect to x_j will again be zero. Inserting Eq.(2.1) into equation (2.2) we obtain

$$\partial \ln P / \partial x_j \;\; = \;\; - \;\; (\ln x_j \; + \; 1) \quad = \;\; \lambda \;\; w_j \qquad (2.3).$$

In this operation for x_j all other variables are kept constant and we may solve Eq.(2.4) for $x = x_j$. The number N(w) of people in income class (w) is given by

$$\mathbf{N\,(w) \;\; = \;\; A \;\;\; exp \, (- \, w \, / \, \langle w \rangle \,)} \qquad \mathbf{(2.4).}$$

In the physical model the relative number of people (x) in the income class (w) follows a Boltzmann distribution, Eq.(2.4). The Lagrange parameter λ has been replaced by the mean income $\langle w \rangle$.

The constant A is determined by the total number of people N_1 with an income following a Boltzmann distribution and may be calculated by the integral from zero to infinity,

$$\mathbf{N_1 \quad = \;\; \int N\,(w)\; d\; w \;\; = A \;\; \langle w \rangle} \qquad \mathbf{(2.5).}$$

The amount of capital K(w) in the property class (w) is

$$\mathbf{K\,(w) \quad = \quad A \;\; w \;\; exp \, (- \, w \, / \, \langle w \rangle \,)} \qquad \mathbf{(2.6).}$$

The total amount K_1 is given by the integral from zero to infinity,

$$\mathbf{K_1 \quad = \;\; A \;\; \int w \; x\,(w) \; d\; w \;\; = \;\; A \;\; \langle w \rangle^2} \qquad \mathbf{(2.7).}$$

The ratio of total wealth (2.7) divided by the total number of households (2.6) leads to

$$K_1 / N_1 = \langle w \rangle \qquad (2.8).$$

And is indeed the mean income <w> per household.

The Lorenz distribution y = K(w) as a function of x = N(w) – fig.1 – may be calculated from the Boltzmann distribution. Eqs.(2.6) and (2.4) and leads to

$$y = x + (1 -) \ln (1 - x) \qquad (2.9).$$

2.1 German wealth data 1993 and the Boltzmann distribution

The Boltzmann distribution in Eqs. (2.5) and (2.6) may be compared to the property data in table 1.

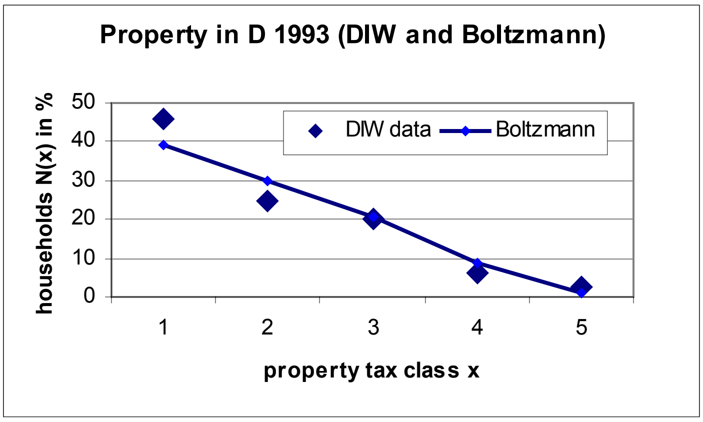

Fig. 2 *Distribution of households x(w) in property classes (w) in Germany 1993, data points according to DIW (14), curve according to Boltzmann, Eq.(2.4). Due to the data set of uneven classes of wealth in table 1 the function does not look like a Boltzmann distribution.*

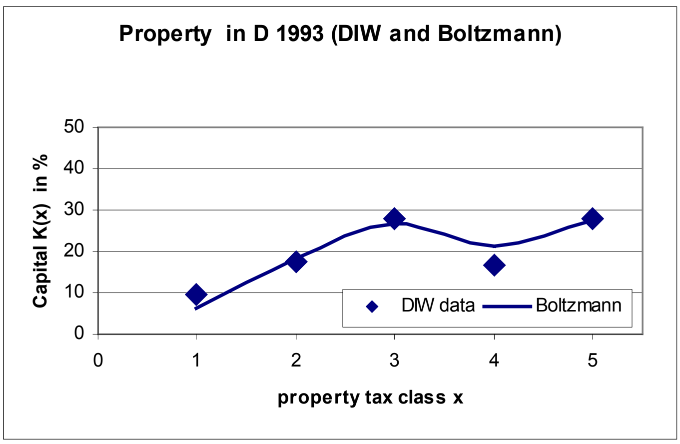

Fig. 3 *Distribution of capital K(w) in property classes (w) in Germany 1993, data points according to DIW (14), curve according to Boltzmann, Eq.(2,6) Due to the data set of uneven classes of wealth in table 1 the function does not look like a Boltzmann distribution.*

Calculations and data show a good agreement:

1. The data of the Lorenz distribution in fig 1 agree well with the calculation of Eq.(2.9).
2. The data for N(w) in fig. 2 agree rather well with Eq.(2.4) with small deviations.
3. The data for K(w) in fig. 3 also agree rather well with Eq.(2.6) again with small deviations.
4. The total number of persons is determined by the integral (from A = 0 to B = ∞)

$$\int N(w)\, dw = A_0 \langle w \rangle = N_0 = 19 \text{ Mill Hh} \qquad (2.5).$$

   The integral leads to the correct number of total households and is used to calculate A_0.

5. The total amount of property is determined by the integral (from A = 0 to B = ∞)

$$\int K(w)\, dw = A_0 \langle w \rangle^2 = K_0 = 190 \text{ Mrd. DM} \qquad (2.7).$$

   The integral leads to the correct amount of total capital

6 The Lagrange parameter $\lambda = 1 / \langle w \rangle$ is determined by the ratio of total property and total number households,

$$\langle w \rangle = K_0 / N_0 = 10.000 \text{ DM} \qquad (2.8).$$

This is the mean property of all households in Germany 1993.

2.2 US wages data 1995 and the Boltzmann distribution

Wages like wealth may be expected to show a Boltzmann distribution according to eq.(2.4). However, jobs below a wage minimum w_0 have the attractiveness $a^* = 0$. Accordingly, the distribution of income will be given by

$$\mathbf{N(w) = a^*(w, w_0)\ \exp(-w / \langle w \rangle)} \qquad \mathbf{(2.10).}$$

The number of people earning a wage (w) will depend on the job attractiveness $a^*(w, w_0)$ and the Boltzmann function. Low wages will be very probable, high wages less probable. Figs. 4 and 5 show the wage distribution for service and manufacturing in the US in 1995 (12).

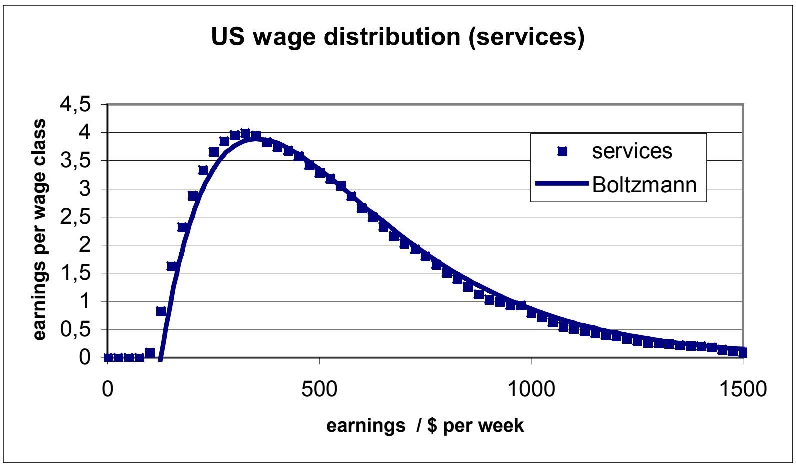

Fig. 4 *Number of people in wage classes in services in the US (15). The data have been fitted by the Boltzmann distribution, ,Eq.(2.10) with* $a^*(w, w_0) = (w - w_0)$.

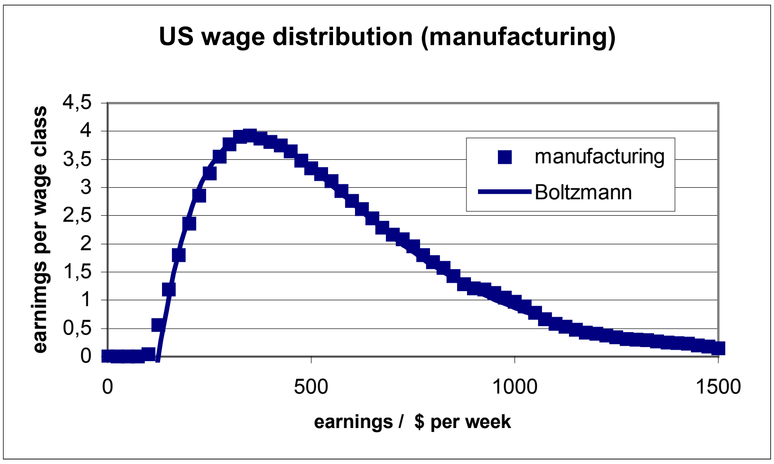

Fig. 5 Number of people in wage classes in manufacturing in the US (15) .The data have been fitted by the Boltzmann distribution , Eq.(2.10) with $a^*(w, w_0) = (w - w_0)$.

Again the Boltzmann distribution seems to be a good fit. However, some authors prefer to fit similar data by a log normal distribution e.g. F. Clementi and M. Gallegatti (15). Presently, both functions seem to apply equally well.

3.0 US wealth data 1993 to 2001, Boltzmann and Pareto Distribution

Yakovenko [12] and others have presented data that follow a Boltzmann distribution for the majority of normal wages and a Pareto distribution for the income of the minority of very rich people, fig. 6.

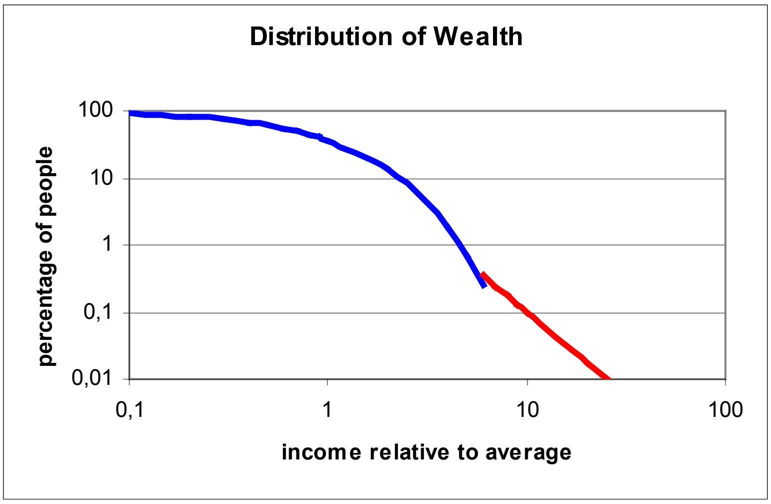

Fig. 6 US income (10): Percentage of people in wage classes relative to average. The wealth of the majority of people (97 %) follows a Boltzmann distribution, only the wealth of the minority of super rich follows a Pareto law. The Pareto tail has a slope between 2 and 3.

3.1 Pareto distribution of wealth

Economic actions (f) are optimized under the constraints of capital, costs or prices (g). Economists are used to maximize the rational production function $U(x_j)$ under constraints of total income $\Sigma\, w_j\, d\, x_j$. The variable (x_j) is relative number of people in each income class (w_j). The Lagrange principle (1.0) is now given by

$$\mathbf{U(x_j) - \lambda\ \Sigma_j\, w_j\, x_j \rightarrow\ maximum!} \qquad \mathbf{(3.0).}$$

At equilibrium (maximum) the derivative of equation (3.0) with respect to x_j will be zero,

$$\partial U / \partial x_j = \lambda\ w_j \qquad (3.1)$$

In economic calculations a Cobb Douglas type Ansatz for the production function U is often applied,

$$\mathbf{U(x_j) = A\,\Pi\ x_j^{\alpha j}} \qquad \mathbf{(3.2).}$$

A is a constant, the exponents α_j are the elasticity constants. Inserting Eq.(3.2) into equation (3.1) we obtain

$$\partial U / \partial x_j = \alpha_j\, A\, x_j^{\alpha j - 1} = \lambda\ w_j \qquad (3.3).$$

In this operation for x_j all other variables are kept constant and we may solve Eq.(3.3) for $x = x_j$. The relative number x(w) of people in income class (w) is now given by

$$\mathbf{N(w) = A\,(\lambda\ w /\ \alpha\, A)^{1/(\alpha - 1)} = C\,(w_m / w)^{2+\delta}} \qquad \mathbf{(3.4).}$$

According to this economic model the number of people N(w) in the income class (w) follows a Pareto distribution! In Eq.(3.4) the Lagrange parameter λ has been replaced by the minimum wealth class of the super rich, $w_m = 1 / \lambda$, the constants have been combined to C. The Pareto exponent stands for $2 + \delta = 1 / (1- \alpha)$. The relative number of people x (w) decreases with rising income (w).

The total number of rich people N_2 with an income following a Pareto distribution is given by the integral from a minimum wealth w_m to infinity,

$$\mathbf{N_2 = \int N(w)\, d\, w = C\ w_m / (1 + \delta)} \qquad \mathbf{(3.5).}$$

The minimum wealth w_m is always larger than zero, $w_m > 0$. The amount of capital K(w) in the property class (w) is

$$\mathbf{K\,(w) = N(w)\, w = C\ w_m\ (w_m / w)^{1+\delta}} \qquad \mathbf{(3.6).}$$

The total amount of capital K_2 of very rich people with an income following a Pareto distribution is given by the integral from a minimum wealth w_m to infinity,

$$\mathbf{K_2 = \int N(w)\, w\, d\, w = C\ w_m^{\ 2} /\ \delta} \qquad \mathbf{(3,7).}$$

For positive wealth the exponent δ needs to be positive, $\delta > 0$. For a high capital of the super rich the exponent δ is expected to be: $0 < \delta < 1$.

The ratio of total wealth (3.7) divided by the total number of super rich households (3.6) leads to

$$\mathbf{w_m = (K_2 / N_2)\ \delta / (1 + \delta)} \qquad \mathbf{(3.8).}$$

3.2 Boltzmann and Pareto distribution in USA wealth data 1993 to 2001

The calculations of the Boltzmann and Pareto law may both be applied to the USA distribution of wealth in fig. 6.

A. Boltzmann distribution

1. The wealth of the majority of the population follows the Boltzmann distribution in Eq.(2.4).
2. The total number of normal rich people is N_1 = 97 % of the total population,

$$N_1 = \int N(w)\, d\, w = A <w> = 0{,}97 \qquad (2.5)$$

3. The wealth of the normal rich population is given by

$$K_1 = \int N(w)\, w\, d\, w = N_1 <w> = 0{,}97 <w> \qquad (2.7)$$

B. Pareto distribution

1. The wealth of the super rich minority follows a Pareto law, Eq.(3.4)
2. The exponent of the Pareto tail in fig. 6 is between 2 and 3 or $0 < \delta < 1$, as required by Eq.(3.7).
3. The minimum wealth of the super rich according to fig. 6 is about eight times the normal mean,

$$w_m = 8 <w>.$$

4. The total number of super rich minority is N_2 = 3 % of the total population,

$$N_2 = \int N(w)\, d\, w = C\, w_m = 0{,}03 \qquad (3.5).$$

5. The total capital of the super rich minority (for $\delta = 0{,}5$) is

$$K_2 = N_2\, w_m\, (1 + \delta) / \delta = 0{,}03 * 8 <w> * 3 = 0{,}72 <w> \qquad (3.8).$$

6. The mean wealth of the super rich minority is

$$<w_2> = (K_2 / N_2) = w_m\, (1 + \delta) / \delta = 8 <w> * 3 = 25 <w> \qquad (3.8).$$

7. In fig. 6 the super rich minority (3 % of the population) owns 40 % (0,72 <w>) of the national wealth and the normal rich majority (97 % of the population) owns 60 % (0,97 <w>).

4.0 Boltzmann and Pareto phases of wealth distribution

The normal rich majority and the super rich minority belong to two different states or phases. The majority is governed by the Boltzmann law, the minority by a Pareto law. This corresponds to two different phases, like liquid and gas in physical sciences, and may be calculated by the Lagrange principle,

$$L = f - \lambda\ g \rightarrow \text{maximum!} \qquad (1.0).$$
$$L_1 = -\, x \ln x - <w> \Sigma_j\, w_j\, x_j \rightarrow \text{maximum!} \qquad (2.0).$$
$$L_2 = A\, x^{\alpha} - <w> \Sigma_k\, w_k\, x_k \rightarrow \text{maximum!} \qquad (3.0).$$
$$y = b - x\ m \qquad (4.0).$$

Eq.(1.0) is the general Lagrange equation, Eq.(2.0) the Lagrange principle in stochastic systems and Eq.(3.0) the Lagrange principle in rational systems. All may be considered linear equations of <w>. The b - value in Eq.(4.0) is given by the entropy or utility function, which is lower for the super rich population due to the small factor A, which is of the order of A = 0,1 in fig.6. The slope "m" is given by the total wealth $\Sigma_j\, w_j\, x_j$, which is higher for the normal population (60 %).

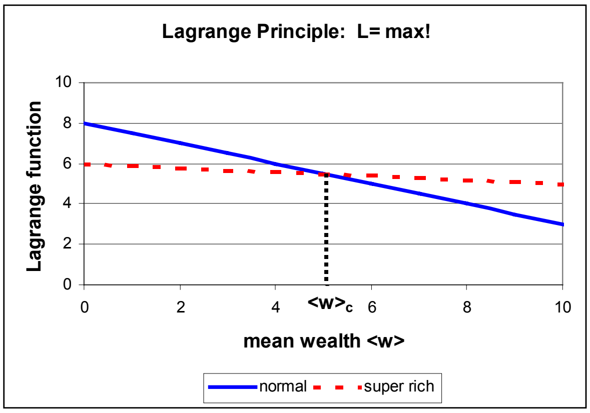

Fig. 7. *For low mean income <w> the Lagrange function L_1 (solid line) is at maximum, the normal rich phase dominates. For very large mean income <w> the Lagrange function L_2 (broken line) is higher (at maximum), the super rich phase dominates.*

The borderline between the normal and the super rich population is given by the intersection of the two lines at $\langle w \rangle_c$ in fig.7. Below $\langle w \rangle_c$ the solid line is higher (at maximum) and the normal phase dominates. Above $\langle w \rangle_c$ the broken line is higher (at maximum) and the super rich phase dominates. The transition point $\langle w \rangle_c$ is given by $L_1 = L_2$. However, the data are not yet sufficient to tell whether the transition "normal" – "super rich" really is of first order, as it is indicated by the sharp knee in fig. 6 and the intersection in fig. 7. Other authors (Yoshi) find a smooth second order transition from the Boltzmann region of normal people to the Pareto region of the super rich. The point of transition is important for the full understanding of the system, but even more important is the mechanism that keeps normal and super rich people separated and drifting more and more apart. This topic will be discussed in a separate paper on the mechanism of economic growth.

Conclusion

What is the advantage of introducing the Lagrange function to economics? The Lagrange function leads to a better understanding of economic interactions:

1. Most economic systems may be regarded to be stochastic with a probability that depends on costs and prices. Normal wealth and income are a matter of probability, as are selling, buying, finding a job or getting married.
2. The Lagrange principle includes stochastic and rational views and will also allow the inclusion of other probability theories (eg. game theory).
3. The Lagrange function links economic systems to social and natural systems such as physics, chemistry, meteorology, which are also governed by the Lagrange principle. This is one reason why many natural scientists are now involved in econo-physics and socio-economics.
4. As in other scientific disciplines, the Lagrange principle may be regarded as the basis of economics, and all economic properties of the system should be calculable from the Lagrange principle! However, the Lagrange principle does not include time. For time dependent properties other methods like. the Fokker Planck or Master equation (3) need to be applied.

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