Quantum simulation via all-optically generated tensor network states

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We devise an all-optical scheme for the generation of entangled multi-mode photonic states encoded in temporal modes of light. The scheme employs a nonlinear downconversion process in an optical loop to generate one- and higher-dimensional tensor network states of light. We illustrate the principle with the generation of two different classes of entangled tensor network states and report on a variational algorithm to simulate the ground-state physics of many-body systems. We demonstrate that state-of-the-art optical devices are capable of determining the ground state properties of the spin-1/2 Heisenberg model. Finally, implementations of the scheme are demonstrated to be robust against realistic losses and mode mismatch.

General quantum states possess a complex entanglement structure that makes their description on a classical computer inefficient in the sense that generally the computational effort grows exponentially with the number of subsystems. However, in ground and thermal states of local Hamiltonians the entanglement and correlations are typically more limited as they satisfy area laws [1–3]. Such states can be approximated well in terms of matrix product states (MPS) or, more generally, tensor network (TN) states parameterization, in which only a polynomial (in the number of subsystems) number of parameters is required to describe the state [4, 5]. This class includes not only the ground states of a wide variety of quantum many-body Hamiltonians [2, 6] but also eponymous examples of entangled states such as the cluster states, GHZ state and W state. Although matrix product states can be efficiently manipulated on a classical computer [7], the treatment of TN states in higher spatial dimensions remains challenging as the computational effort, whilst polynomial, grows with a high power in the number of subsystems and bond dimension. Therefore, the experimental generation of TN states and their use for quantum simulation is of considerable interest.

Current experimental implementations for the generation and processing of TN states focus on spatial modes of light but these implementations require experimental resources that typically increase quickly with the required size of the TN state [8–12]. This limitation can be overcome by using the temporal modes of light or time bins, which provide an infinite-dimensional Hilbert space that can be controlled with constant experimental resources through time multiplexing. The potential of this approach has already been successfully demonstrated in the context of quantum walks and boson sampling [13–16]. Existing proposals for generating photonic TN states in the temporal modes of light rely on the strong coupling of light to a single atom trapped inside a cavity [17–19]. The strength of these methods is that they allow the generation of arbitrary 1D TN states whose entanglement is limited only by the number of accessible atomic states. However, the experimental implementation of these schemes requires two challenging conditions to be met, namely the cooling and localizing of the atom, and strong coupling between the atom and the light emitted from the cavity. Moreover, the requirement of complete control over multiple atomic states restricts the amount of entanglement in the generated TN states.

In this work, we devise an all-optical scheme for the generation of TN states in one and higher dimensions that overcomes these challenges. Our scheme does not suffer from the stringent requirement of strong atom–photon coupling and instead exploits well established parametric downconversion (PDC) methods to build entanglement.
in the generated state [20]. Furthermore, our method overcomes the restriction on entanglement (as quantified by bond dimension) to accessible atomic levels by using the photon-number degree of freedom to share entanglement between components of the generated state. Finally, our all-optical scheme also promises robustness against loss and mode mismatch and can be realized with current optical technology.

We first describe the relevant scheme that generates entangled one- and two-dimensional TN states. We devise a state-generation procedure for TN states including but not restricted to W and the GHZ states in temporal modes of light, thus going beyond current schemes that can generate these states in spatial modes of light [8, 21–23]. Finally, we present a variational algorithm to simulate the ground-state physics of quantum many-body systems. We demonstrate that the scheme can be used to simulate the ground-state physics of the spin-1/2 Heisenberg model using current optical technology with realistic assumptions on experimental constraints such as losses, mode-mismatch and limited number of measurements.

**Scheme to generate TN states** — Our proposed scheme to generate entangled multi-mode states of light is depicted in FIG. 1. The experimental setup relies on placing a PDC nonlinearity into an optical loop and optically pumping the nonlinearity. This nonlinearity performs two-mode squeezing $U = \exp(\eta a_1^\dagger a_2^\dagger - \eta^* a_1 a_2)$ on the horizontal and vertical modes of light, where the PDC parameter $\eta$ depends on the strength of the optical pumping. Here $a_i^\dagger$ and $a_i$ are the creation and annihilation operators for mode $i$. The light in one of the two polarization modes (say vertical) is coupled out of the loop via a polarizing beam-splitter (PBS), while the other (say horizontal) cycles the loop. An electro–optic modulator (EOM) in the loop dynamically mixes the two polarization modes, of which the vertical mode is in vacuum, via arbitrary linear transformations $\hat{a}_i \rightarrow \sum_{i=1}^{2} V_{ij} \hat{a}_i$ for $2 \times 2$ special unitary matrix $V \in SU(2)$ [24]. The time it takes for light to cycle the loop is set equal to the delay between subsequent pump pulses. Thus, the cycling light arrives synchronous to the next pump pulse and effects two-mode squeezing interaction between the two polarization modes [25]. In other words, the PDC and the EOM together give rise to an interaction between the horizontally-polarized cycling light and the vertically-polarized optical vacuum.

The quantum circuit representing this repeated interaction is presented in FIG. 2. We consider the temporal modes of the light coupled out from the loop over many cycles. We show the establishment of multi-particle entanglement between subsequent temporal modes mediated by the light cycling in the loop as depicted by the dashed line of FIG 2. Specifically, we show that the emitted temporal modes of light permit a 1D TN representation and include entangled states such as W and GHZ states. The proof for this result and the general form of the resultant TN state is in the SI. The intuition for the proof is that the cycling mode mediates entanglement between subsequently emitted light modes. Entanglement between one emitted mode and the next is limited by the entanglement between the first mode and the cycling mode, and this maximum entanglement is constant irrespective of the number of cycles. Because subsequent temporal modes of light are entangled, albeit with limited entanglement, it follows that the state of the emitted light can be represented as a TN state of limited bond dimension.

Although the properties of 1D TN states can be efficiently obtained on a classical computer, those of TN states in two and higher dimensions require classical algorithms that scale badly, i.e., exponentially in the system size and as high-degree polynomials in the bond dimension. In other words, two and higher-dimensional TN states can be exploited for obtaining non-trivial quantum-computational speedup. It is possible to modify our
scheme to generate higher-dimensional TN states, by connecting additional optical loops into the existing loop as depicted in FIG. 3. The effect of one additional loop is to convert different polarization modes into temporal modes, an approach already used in 2D quantum walks [13, 26]. Optionally, additional nonlinearities and EOM can be added to the loop to ensure that the entanglement structure is identical in the two dimensions of the lattice. The additional optical loop is designed to provide a time delay of $\tau/n$, which is smaller than the cycling time $\tau$ of the main loop by a factor $n$ for some large integer $n$. Owing to this additional time delay, the difference between the emission times of two temporal modes is either $\tau$ or multiples $2\tau/n, 3\tau/n, \ldots$ of the interval $\tau/n$. Modes with time difference $\tau/n$ are interpreted as neighbors along one axis of the TN lattice whereas those with time difference $\tau/n$ are interpreted as neighbors along a different axis. Depending on the required number $n$ of lattice sites, one can choose any $n > \tilde{n}$ so that the sites in the 2D lattice are uniquely defined; values up to $n = 62$ have been demonstrated in recent photonic quantum-walk demonstrations [13, 27]. Thus, the emitted light possesses an entanglement structure that is captured by a 2D TN state with a triangular structure (See SI). Similarly, additional loops can be connected to the optical setup to generate higher-dimensional TN states.

State-generation — Here we detail how the setup can be used to generate two inequivalent classes of entangled states, namely the W state and the GHZ state. First, we consider the $m$-qubit W state $|\psi\rangle_w = |0\ldots01\rangle + |0\ldots10\rangle + \cdots + |1\ldots00\rangle$, which has one excitation $|1\rangle$ that is delocalized uniformly over all the qubits. Our proposed setup can generate a heralded W state, which is defined as

$$|\psi\rangle_w = |0\ldots00\rangle \otimes |0\rangle + \eta \left( |0\ldots01\rangle + |0\ldots10\rangle + \cdots + |1\ldots00\rangle \right) \otimes |1\rangle \quad (1)$$

on a total of $m + 1$ qubits for some complex $\eta$ with $\eta < 1$ and the normalization factor is omitted for simplicity. In this state, a $|1\rangle$ in the last qubit heralds the presence of a W state in the remaining qubits whereas a $|0\rangle$ in the last qubit implies a vacuum state in the remaining qubits.

The heralded W state can be generated by our proposed setup in the single-rail basis [28], wherein the absence of a photon in a temporal mode encodes the state $|0\rangle$ and a single photon in the mode encodes $|1\rangle$. Cases where more than a single photon is present in the mode are discarded, and the probability amplitude of such cases can be made arbitrarily small by choosing a suitably small pump strength. The generation of W states can be accomplished by using the same pump pulse strength in each cycle, i.e., by setting $V^{(i)} = 1$ and $U^{(i)} = U$ over each cycle $i$ except the last cycle, in which the EOM effects a swap $V^{(m)} = S$ and the pump is turned off $U^{(m+1)} = 1$. These parameters lead to an excitation $|1\rangle$ that is in a superposition over $m$ different sites (See SI for details).

The relative weights of the different components in the state of EQ. (1) can be adjusted by varying the strength of the pumping and thereby the interaction strength $\eta_i$ in each cycle $i$.

Next we describe the generation of the four-qubit GHZ state $|\psi\rangle_{\text{GHZ}}$, which is usually defined as an equal superposition $|0000\rangle + |1111\rangle$ over each qubit that is in state $|0\rangle$ and each qubit in state $|1\rangle$. An alternative description of the GHZ state $|1100\rangle + |0011\rangle$ is obtained by redefining the qubit labels in the last two qubits. Our proposed setup can be used to generate the diluted GHZ state

$$|\psi\rangle_{\text{diluted}} = |0000\rangle + \eta \left( |0011\rangle + |1100\rangle \right), \quad (2)$$

where the normalization factor is omitted. To obtain this state, the pump and the EOM are turned on in the first and third cycle and turned off in the second and fourth cycle, i.e., $U^{(i)} = V^{(i)} = I$ for even $i$ and $U^{(i)} = U$, $V^{(i)} = S$ for odd $n$. This creates two excitations in a superposition over the first two and the last two temporal modes, which is the required diluted GHZ state (see SI). The relative weights of the two terms can be adjusted by varying the pumping strength. Thus, the proposed setup can be used to generate two inequivalent classes of entangled states, namely the W state and the GHZ state, just by reprogramming the EOM and the pump control.

Simulations provide evidence that our state generation procedure is robust against the usual experimental imperfections of loss and mode-mismatch from the PDC. Consider reasonable experimental losses which are typically upwards of 10% loss in each cycle; these losses can lead to higher than 90% fidelity with respect to target state as seen in the red and green dots of FIG. 4.

Quantum variational algorithm — Other than state preparation, the proposed setup can be exploited for performing a mixed quantum-classical algorithm for the determination of ground state properties of many-body systems via a quantum variational approach, which we now describe. We consider the task of determining the properties, such as the energy or correlations, of the ground state of a given Hamiltonian operator that acts on qubits. The generated TN states comprise the set of variational states; their energy with respect to the given Hamiltonian is obtained by performing Glauber correlation measurements on the output light following the procedure of [30]. Measurements on output light can be performed via standard techniques in single-rail representation [31, 32]. A classical minimization algorithm can then be used to obtain circuit parameters corresponding to the generated state that has lowest energy with respect to the given Hamiltonian. If the circuit parameters, including the pump strength and EOM parameters, are sufficiently expressive, i.e., if the ground state is close to the class of variational states generated by the setup, then an accurate approximation of the given Hamiltonian’s ground state can be obtained. The procedure is
expected to work well for a wide variety of Hamiltonians because the ground state of most 1D local Hamiltonians is close to a low-dimension TN state [2]. The properties of the ground state can be determined by usual measurements on the output light. Our mixed quantum-classical variational approach encompasses the variational problem that can be solved using the so-called Ising machines because it exploits the polarization and photon-number degrees of freedom in addition to temporal modes used in Ising machines [33, 34].

To illustrate the performance of this approach, we simulate the procedure to find the ground state of the isotropic XY model [35, 36]. The ground state of XY Hamiltonian $H_{\text{XY}} = J \sum_i X_i X_{i+1} + Y_i Y_{i+1} + B \sum_i Z_i$ is the W state for certain range of $B$ [37]. We simulate Glauber correlation measurements on the output light to obtain the energy of the generated state for a specific value of circuit parameters [30]. Starting with random circuit parameters, we use a constrained minimization algorithm to find those circuit parameters that minimize the energy. The variational minimization returns a state that is close to the expected ground state as depicted in FIG. 5. Simulations provide evidence that this approach is robust against statistical noise (FIG. 5b), i.e., that our scheme performs well even for limited numbers of experimental measurements. FIG. 5c presents simulation results for robustness against loss including coupling inefficiencies, optical loss in the loop medium and the effect of mode mismatch. Simulations indicate high output fidelities even for realistic losses of around 30 percent.

A similar variational approach can also be used to enhance the quality of state generation, as described above, against possible experimental imperfections. For instance, consider the task of improving the fidelity $F = \langle \psi_{\text{lab}} | \rho_{\text{lab}} | \psi_{\text{lab}} \rangle$ of the generated state $\rho_{\text{lab}}$ with respect to a given target state $| \psi_{\text{lab}} \rangle$, such as the W state, under the presence of imperfections such as loss and phase drift. We can leverage from a measurement-based feedback control scheme [38] to find the circuit parameters that maximize fidelity against a desired state and thereby compensate for experimental imperfections. Direct fidelity estimation procedures [39, 40] can be used to efficiently estimate the fidelity with respect to the desired state and classical optimization can be performed to maximize this fidelity.
Simulations show that our W and GHZ state generation procedures can be made further resilient to loss via such feedback control by two-three orders of magnitude (see blue and green crosses in FIG. 4).

**Conclusion and discussion** — In summary, we propose a scheme for the all-optical generation of one- and higher-dimensional TN states in temporal modes of light in the single-rail basis. This scheme enables generating inequivalent types of entangled states and is robust against realistic losses and mode mismatch.

The free parameters describing the TN state and its bond dimension can be enhanced by using additional degrees of freedom of light such as spatial modes, time-frequency Schmidt modes and orbital angular momentum modes of light [41–44]. The number of free parameters can be increased by adding another EOM to the loop between the PDC and the PBS. Finally, states such as coherent states can be impinged into the PDC instead of starting with the optical vacuum, thereby leading to the generation of high-photon-number Gaussian matrix product states [45, 46], which could potentially be used as a resource for Gaussian boson sampling [47, 48].

The fidelity of the generated states with respect to the expected TN states can be estimated directly and efficiently, i.e., using measurement and processing time that scales only polynomially in system size [39, 40]. Furthermore, because the generated states are TN states, efficient TN-based procedures can be employed to perform tomography of the states [49–52].

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SUPPLEMENTARY INFORMATION

A. Modeling the setup

Here we provide details about the modeling of the setup and the generation of W and GHZ states as described in the main text. First, we describe our modeling of the PDC and the EOM in the setup. The PDC performs two-mode squeezing on the light in the two orthogonal polarization modes. We model the downconversion process by the Hamiltonian

$$\hat{H} = \int d\omega_1 d\omega_2 F(\omega_1, \omega_2) \left( \kappa \hat{a}_1^\dagger \hat{a}_2^\dagger + \kappa^* \hat{a}_1 \hat{a}_2 \right),$$

(3)

where $\hat{a}_1$ and $\hat{a}_2$ are the photon annihilation operators corresponding to the two polarization modes; the spectral amplitude $F(\omega_1, \omega_2)$ depends upon properties of the pump and the nonlinearity; and $\kappa$ is the complex-valued interaction strength [53, 54]. We assume no spectral correlations and perfect mode matching. While the latter of these two imperfections can be accounted in terms of additional loss in the loop, the former can be modeled by considering dilution with a fully-mixed state [55, 56]. Under these assumptions, the analysis simplifies to monochromatic photons and the Hamiltonian to

$$H = \kappa \hat{a}_1^\dagger \hat{a}_2^\dagger + \kappa^* \hat{a}_1 \hat{a}_2.$$  

(4)

Note that the interaction strength $\kappa$ depends on the pump power, which can be adjusted independently for each cycle. The unitary transformation effected by this Hamiltonian is given by

$$U = \exp(-i\hat{H}t)$$

$$= \exp(\eta \hat{a}_1^\dagger \hat{a}_2^\dagger - \eta^* \hat{a}_1 \hat{a}_2)$$

(5)

where the magnitude $|\eta|$ of the complex PDC parameter

$$\eta \overset{\text{def}}{=} -i\kappa t$$

(6)

expresses the strength of interaction between the two polarization modes.

The EOM placed in the optical loop allows changing the phase and the polarization of the incoming field with
each cycle. Thus, the EOM effects a beamsplitter-like transformation

\[ \hat{a}_j^\dagger \rightarrow \sum_{i=1}^{2} V_{ij} \hat{a}_i^\dagger \]  

(8)
on the light in the two polarization modes for \(2 \times 2\) unitary transformation \(V\). The PDC parameter \(\eta\) and elements of the SU(2) transformation matrix \(V_{ij}\) comprise the set of free parameters in the generation of the state.

**B. Proof that setup generates TN states**

Our claim is that the light emitted from the optical loop is represented in the photon-number basis as a TN state of low bond dimension. To justify this claim, we consider the state of the vertically-polarized emitted light, while the horizontally polarized light continues to cycle in the optical loop. The intuition for the proof is that the entanglement between any two subsequent modes is mediated only by the cycling light, which places a limitation on entanglement between any two partitions of the emitted state. Figure 6 depicts this intuition and the structure of the generated TN states.

In the photon number basis, the transformations on the two polarization modes in the \(i\)-th cycle comprise a two-mode squeezing \(U^{(i)}\) operation on the cycling and the emitted light, which leads to the light in the cycling mode emitted from the setup and thus decouples the cycling mode from the emitted entangled state. Thus the state of the emitted \(m\) modes is

\[ T^{(m-1)j_m-1}\cdots T^{(2)j_2}{T^{(1)j_1}} |0j_1j_2\cdots j_{m-1}\rangle \]  

(14)

where the cycling mode is in vacuum state and is decoupled from the emitted modes. Identically, the emitted state can be expressed as

\[ T^{(m-1)j_m-1}\cdots T^{(2)j_2}{T^{(1)j_1}} |0j_1j_2\cdots j_{m-1}\rangle \]  

(15)

The state (15) is the sum of basis states \(|j_1j_2\cdots j_{m-1}\rangle\) with coefficients in the form of a product of \(D\)-dimensional matrices \(T^{(n)j_n}\); states in the form are matrix product states (MPS), or equivalently 1D TN state, of bond dimension \(D\). This completes our proof. Furthermore, Eq. (15) also gives a general parameterization of the states that can be generated via our scheme.

**C. Generation of W and GHZ states**

While our TN state generation procedure can generate a broad class of TN states parameterized by the EOM parameters and the PDC interaction strength \(\eta\) in each loop, we focus on two classes of entangled states for concreteness. Here we show how we can generate the W and GHZ states using the scheme.

*Generation of W state* — Here we illustrate the generation of W states, which admit an MPS representation with bond dimension \(D = 2\) [57, 58]. We calculate in the single-rail basis and truncate our Hilbert space to no more than one photon in each mode, i.e., we neglect states with two or more photons in any of the modes. Equivalently, we ignore probabilities of order \(\eta^4\) and higher powers, where \(\eta\) is the PDC interaction parameter (7).
FIG. 6. Quantum circuits to generate (a) one- and (b) 2D TN states. The vertical blue-green rectangles represent EOM and PDC transformations between two polarization modes. The horizontal transformations represent the action of EOM, PDC and the fiber-loop on polarization and temporal modes (See text). The filled circles represent emitted modes while the filled black circled represent modes that continue to cycle in the loop. A dashed brown edge between two modes represents correlations introduced by the circuit between the modes. The overall structure of these modes and their connecting edges defines the structure of the TN. The labels in (b) are omitted for simplicity.

The generation of W states can be accomplished by setting $V = 1$, i.e., by programming the EOM in the optical loop to leave the light unchanged, and by using the same pump pulse strength in each cycle. For low pump strength, we can truncate the Taylor expansion of the unitary transformation (6) to low orders in $\eta$ as

$$U = \exp(\eta a_1^\dagger a_2^\dagger - \eta^* a_1 a_2)$$

$$\approx 1 + (\eta a_1^\dagger a_2^\dagger - \eta^* a_1 a_2) + \frac{1}{2} \left( \eta^2 a_1^\dagger a_1 a_2^\dagger a_2^\dagger + \eta^2 a_2^\dagger a_2 a_1^\dagger a_1 \right) + \ldots$$

$$= 1 + (\eta a_1^\dagger a_2^\dagger - \eta^* a_1 a_2) + \frac{1}{2} \left( \eta^2 a_1^\dagger a_1 a_2^\dagger a_2^\dagger + \eta^2 a_2^\dagger a_2 a_1^\dagger a_1 - 1 \right) + \ldots,$$

which is correct up to the first two orders in $\eta$.

For calculations of the light state, we adopt the notation introduced in the main text where the first mode refers to the cycling horizontally polarized mode and the remaining modes are the emission modes arranged in increasing order of emission. For concreteness, we calculate the state of light at the end of four cycles. The starting state $|0000\ldots\rangle$ comprising vacuum in each of the modes is transformed by the PDC to

$$|\psi_1\rangle = U^{(1)} |0000\ldots\rangle$$

$$= |0000\ldots\rangle + \eta |1100\ldots\rangle + \eta^2 |2200\ldots\rangle,$$  

where the subscript in $|\psi_1\rangle$ denotes the number of completed cycles, and we have omitted the $(1 - \eta^2)^{1/2}$ normalization factor for simplicity. Next, $U$ acts on the first and third modes while the remaining modes are left unchanged

$$|\psi_2\rangle = U^{(2)} |\psi_1\rangle$$

$$= |0000\ldots\rangle + \eta |1010\ldots\rangle + \eta^2 |2020\ldots\rangle + \ldots$$

$$+ \eta |1100\ldots\rangle + \sqrt{2}\eta^2 |2110\ldots\rangle + \ldots$$

$$+ \eta^2 |2200\ldots\rangle + \ldots,$$

where terms up to two orders in $\eta$ are considered. Then, $U$ acts on the first and fourth modes while the remaining modes are left unchanged

$$|\psi_3\rangle = U^{(3)} |\psi_2\rangle$$

$$= |0000\ldots\rangle + \eta |1001\ldots\rangle + \eta^2 |2002\ldots\rangle + \ldots$$

$$+ \eta |1010\ldots\rangle + \sqrt{2}\eta^2 |2110\ldots\rangle + \ldots$$

$$+ \eta^2 |2200\ldots\rangle + \ldots$$

The terms of each order in $\eta$ can be grouped to obtain

$$|\psi_3\rangle = |00\ldots\rangle + \eta |1\rangle \otimes (|001\ldots\rangle + |010\ldots\rangle + |100\ldots\rangle)$$

$$+ \eta^2 |2\rangle \otimes (|200\ldots\rangle + |202\ldots\rangle + |020\ldots\rangle + |002\ldots\rangle) + \ldots$$

$$+ \sqrt{2}\eta^2 |2\rangle \otimes (|011\ldots\rangle + |101\ldots\rangle + |110\ldots\rangle) + O(\eta^3).$$

In the last cycle, the EOM performs the swap operation and the pump is turned off, thereby allowing the light in the cycling mode to exit the loop. Thus, the final state is given by

$$|\psi_4\rangle = |0\rangle \otimes |000\rangle + \eta |1\rangle \otimes (|001\rangle + |010\rangle + |100\rangle),$$

which has been truncated to first order in $\eta \ll 1$. Writing the output state in the order of the departure of the photons, we have the unnormalized state

$$|\tilde{\psi}_4\rangle = |000\rangle \otimes |0\rangle + \eta (|001\rangle + |010\rangle + |100\rangle) \otimes |1\rangle,$$

which is interpreted a heralded W state as follows. If no photon is emitted out of the loop in the last cycle, (i.e., the last departure time mode), then only vacuum was be emitted in the preceding three modes (first three cycle). On the other hand, with probability $\eta^2$, a single photon will be emitted in the last cycle, and this emission
‘heralds’ (or announces belatedly) the emission of a W state
\[ |\psi_w\rangle = |001\rangle + |010\rangle + |100\rangle \] (25)
in the preceding three modes. FIG. 7a depicts the quantum circuit representing this scheme.

In general, running the experiment for \( m + 1 \) cycles will produce a heralded \( m \)-mode W state
\[ |\psi_w\rangle = |00\ldots01\rangle + |00\ldots10\rangle + \cdots + |10\ldots00\rangle . \] (26)
Note the heralding photon is emitted after each of the other modes, so experiments that require the W state can employ post-selection to choose only that subset of data in which a photon is emitted in the last cycle.

**Generation of four-mode GHZ state** — Another important TN state is the GHZ state. We show here that the setup can be used to generate a four-mode GHZ state. The four-mode GHZ state is given by \(|0000\rangle + |1111\rangle\) or equivalently by \(|0011\rangle + |1100\rangle\). We present a construction of the latter form of the GHZ state here.

Similar to the W state analysis, we calculate in the ordering of arrival time and ultimately switch to departure-time ordering for analyzing the output. The action of the first unitary on the vacuum state gives the unnormalized state
\[ |\psi_1\rangle = U^{(1)} |0000\rangle = |0000\rangle + \eta |1100\rangle + \eta^2 |2200\rangle + \ldots \] (27)
The first EOM operation is set such that it swaps the state of light in its two modes
\[ V^{(1)} = S \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \] (28)
which gives
\[ |\psi_{2'}\rangle = V^{(1)} |\psi_1\rangle = |0000\rangle + \eta |0110\rangle + \eta^2 |0220\rangle + \ldots \] (29)

Next, the light enters the PDC whose pump power is set to zero \((U^{(2)} = I)\) and the EOM action is set as identity \(V^{(2)} = I\). Hence, the state is unchanged \(|\psi_{2'}\rangle = |\psi_{2'}\rangle\) on going through the PDC and the EOM. The vertical polarization mode is coupled out of the optical loop via the PBS; the loop elements will subsequently act on the next temporal mode. Upon action \(U^{(3)} = U\) of the PDC in the next cycle, the state changes to
\[ |\psi_3\rangle = U^{(3)} |\psi_{2'}\rangle = U^{(3)} |\psi_{2'}\rangle = |0000\rangle + \eta |1001\rangle + \eta^2 |0202\rangle \]
\[ + \eta |0110\rangle + \eta^2 |1111\rangle + \eta^2 |0220\rangle + \ldots \] (31)

Ignoring terms containing amplitude \(\eta^2\) and higher powers of \(\eta\), we obtain
\[ |\psi_3\rangle = |0000\rangle + \eta (|1001\rangle + |0110\rangle) \] (32)
As usual, a swap operation at the end of the process leads to the emission of the cycling mode. Arranging the modes in order of departure, the output light is represented by the state
\[ |\psi_3\rangle = |0000\rangle + \eta (|0011\rangle + |1100\rangle) \] (33)
which is a superposition of the vacuum state with the GHZ state. This completes our construction of the GHZ state. We depict the circuit corresponding to this construction in FIG. 7b.

**D. Simulations of experiment**

Here we present relevant details of the simulations for the results presented in FIGS. 4 and 5. The state of light in the setup was represented in the photon number basis using a 1D TN, i.e., a MPS representation because a full density matrix representation of the generated output was not feasible. We used the library mpnum [59] to simulate 1D TNs and basic operations thereon. The Fock space was truncated to no more than three photons in each mode. We constructed the PDC and EOM operators in the photon-number basis representation via python.
package qutip \cite{60} and translated these into a matrix product operator (MPO) form. Similarly, measurements were represented as projection MPOs.

We include the experimental imperfections of loss, mode-mismatch and a finite number of measurements. Loss and mode-mismatch are modeled by assuming an additional beamsplitter in the loop. This beamsplitter is assumed to receive vacuum in one input port and the cycling mode in the other, and one of its output ports is grounded while the other continues as the new lossy cycling mode. Experimental noise is simulated by adding the expectation value calculated from the simulations with Gaussian random noise, i.e., Gaussian random numbers that have mean zero and variance equal to the operator variance divided by the inverse of the number of measurements.

The simulations for the quantum variational algorithm (FIG. 5) are performed as follows. A random set of circuit parameters is chosen as the initial setting of the circuit. Next, measurements are simulated on the state of the system based on the given Hamiltonian. Variational minimization is performed over the circuit parameters to minimize the energy with respect to the given Hamiltonian. The fidelity of the state is computed against the exact target state, which is also represented as an MPS.

Simulations for feedback control (FIG. 4) start with the ideal circuit parameters, i.e., parameters that lead to a correct state generation without loss and other imperfections. Imperfect state generation is simulated as described above. Next, circuit parameters that maximize the fidelity with respect to a given target state are found by variational maximization. The fidelity of the state thus generated is higher by one to two orders of magnitude higher than those obtained without variational optimization. In both these cases, we use the constrained optimization by linear approximation (COBYLA) algorithm to perform constrained numerical optimization \cite{61, 62}.

F. Experimental details

Here we discuss some critical experimental details for the implementation of the all-optical generation of TN states as we show them in FIGS. 1 and 3. In this case critical parameters are the losses in the setup and in case of PDC mode mismatch. With the present technology, a free space loop as it is shown here has almost no losses, whereas a fibre loop introduces losses such as coupling efficiency to the fiber and polarization compensation. The inclusion of a PDC source in the loop, such as a bulk or a waveguide crystal introduces losses and in case of a waveguide probably mode mismatch between the cycling and the PDC mode. Although the setup is robust against losses, they should be minimized in order to reduce the running time of the experiment as well as deviations from the target state.

The last parameter is the source efficiency, its brightness and the coupling efficiency of the generated modes to the loop, which has to be maximized. With the current technology we can have a PDC source with a brightness of $3.8 \times 10^{11}$ photon pairs/(W·s·m) \cite{63}. Taking into account a free space loop transmission of 98%, a mode overlap of 90% and a coupling efficiency of 60% per cycle, a detection efficiency of 85% and a sample length of 8mm, the generation rate of a 4 dimensional Werner or GHZ state is around $\sim 2 \times 10^5$ states per second and per mW of pump power. For reasonable pump powers of 10µW, $\sim 2000$ states per second are generated.