A photonic quantum walk with a four-dimensional coin

Lennart Lorz,1 Evan Meyer-Scott,1 Thomas Nitsche,1 Václav Potoček,2 Aurél Gábris,2,3 Sonja Barkhofen,1,* Igor Jex,2 and Christine Silberhorn1

1Integrated Quantum Optics, Universität Paderborn, Warburger Strasse 100, 33098 Paderborn, Germany
2Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, Břehová 7, 115 19, Praha 1, Czech Republic
3Wigner Research Center for Physics, Hungarian Academy of Sciences, Konkoly-Thege M. u. 29-33, H-1121 Budapest, Hungary

The dimensionality of the internal coin space of discrete-time quantum walks has a strong impact on the complexity and richness of the dynamics of quantum walkers. While two-dimensional coin operators have successfully been used for defining dynamics on complex graphs, higher dimensional coins are necessary to unleash the full potential of discrete-time quantum walks. In this work we present an experimental realisation of a discrete-time quantum walk on a line graph, that instead of a two-dimensional exhibits a four-dimensional coin space. Making use of the extra degree of freedom, we are able to generate quantum walks on cyclic graphs of various sizes and topologies, with mixing and non-mixing coins and different input positions and polarisations. By exploiting the full dimensionality of the coin we additionally demonstrate walk evolutions on figure eight graphs consisting of two cycles connected by a central node of rank four. We implemented the quantum walks via time-multiplexing scheme in a Michelson interferometer loop architecture, employing polarisation and travelling direction of the pulses in the loop as the coin degrees of freedom. The experimental results are supplemented by theoretical analysis of accessible coin operations, plus a scheme to produce arbitrary $4 \times 4$ unitary coin operations.

I. INTRODUCTION

During the past two decades, quantum walks [1–3], the quantum mechanical analogue of random walks, have become an established basis for quantum algorithms [4–8] and quantum simulations [9–13]. Quantum walks have been realised experimentally on various platforms, such as photons [14–23], ions [24, 25], atoms [26–28] and nuclear magnetic resonances [29]. A detailed introduction to experimental implementations of quantum walks can be found in Ref. [30]. Discrete time quantum walks (DTQWs) have been successfully implemented using time multiplexing techniques [18, 19, 31], offering flexibility and easy reconfigurability accompanied by high efficiency and stability. A marked feature of the DTQW is an internal degree of freedom — the coin space — that conditions the spatial shift of the walker, in the same way as a coin toss determines the movement of a classical random walker. It is the dynamics in the coin space that is argued to provide the key ingredient to the complex behaviour of the DTQW [6].

While the initial definition of DTQWs assumed translation invariant and time independent dynamics, more versatility can be obtained by spatial and temporal control of the quantum walk parameters. By varying the coin operation such systems have been used experimentally to observe Anderson localization [21, 23, 32], dynamical localization [26], topological phases [33–40], and other fundamental effects such as recurrence [41] and revivals [42]. The dynamic control of the coin operation can be extended to engineering the topology of the graph on which the walk takes place: finite [31] and percolation graphs [43], and lines with periodic boundary conditions [44] have been demonstrated experimentally.

To have any effect on the walker dynamics, the minimum required dimensionality for the coin space is two. In order to reduce the required theoretical and experimental effort associated with the study of higher dimensional coins, many of the above works employ multi-step protocols, which use only two-dimensional coins. These protocols simulate higher dimensional coins by splitting up each step into multiple coin and shift operations acting on a two-dimensional coin space. They have found use not only for realizing dynamics on graphs embedded in higher dimensions, but also for 1D quantum walks on more sophisticated graphs, such as on percolation graphs or circles [43, 44]. As the required doubling or even triplication of the necessary step numbers for the implementation of such multi-step schemes is experimentally disadvantageous in terms of losses, inaccuracies, and scalability, these protocols significantly impact the efficiency of the physical implementation.

More fundamentally, the two-dimensional coin space is insufficient to reveal all the features of the DTQW even in the case of a one dimensional (1D) graph. As an example, it has been shown that in addition to the Su–Schrieffer–Heeger phases [45, 46] supported by split-step walks, a DTQW with genuine four-dimensional coins can exhibit richer topological phenomenology, i.e. topological phases analogous to the quantum spin Hall phases [45]. For more complex graph topologies (e.g. graphs embedded in higher dimensional spaces, non-uniform rank distribution of nodes) the dimensionality of the coin space is expected to play an even more prominent role, illustrated for example by the Grover walk on a two-dimensional (2D) lattice [47, 48]. The limitations of two-dimensional coins provide
FIG. 1. Experimental realisation of the Michelson geometry. The three $2 \times 2$ polarisation rotations forming the coin are realised by electro-optic modulators (EOM) in combination with either half (HWP) or quarter (QWP) waveplates (WP). We use single mode fibres of 328 m and 338 m length in the arms, the other parts are installed in free space. A waveplate in front of the incoupler mirror determines the input polarization of the pulse. After the outcoupling at the partially reflective mirror the pulses are routed to four superconducting nanowire single photon detectors enabling the resolution of all four internal states.

The structure of the paper is as follows. In Sec. II we present experimental implementations of DTQWs on a line governed by programmably controlled four-dimensional coins, reaching beyond the previous two-dimensional definition and demonstrating QWs on new complex graph topologies. At the heart of our time-multiplexing scheme is an interferometer arranged in a Michelson-type geometry, in contrast to earlier implementations based on a Mach–Zehnder geometry. While offering identical stability and versatility, the present setup introduces a new degree of freedom for the coin, namely the direction of propagation of two counter-propagating optical modes. Together with the polarization they support the four-dimensional coin. The higher-dimensional coin space and the temporal control of the coin operations enable us to efficiently realise DTQWs on non-trivial graphs of different sizes and topologies.

The layout of our experiment, depicted in Fig. 1, resembles a Michelson interferometer closed by a loop. The coherent laser pulse (wavelength 1550 nm) plays the role of the quantum walker, using the mathematical equivalence between wave dynamics and single particle quantum dynamics [49]. The input pulse is coupled into the loop by a beam splitter with low reflectivity $R \approx 1\%$ ensuring high transmittivity for the travelling pulses. The loop allows two propagation directions, clockwise and counter-clockwise, and for each direction we can distinguish two orthogonal polarizations, horizontal and vertical as schematically plotted in Fig. 2b. We label these four orthogonal modes by $cH$, $cV$, $ccH$ and $ccV$. To control the dynamics of the pulses, we insert polarization rotating elements consisting of waveplates and fast-switching electro-optical modulators (EOMs) in the arms $A$ and $B$ as well as in the loop (cf. Fig. 1). The initial pulse with a well defined polarization is coupled into the modes $ccH$ and $ccV$ by the incoupler. The polarization of the pulse is rotated by the waveplate and the EOM before it reaches the polarizing beam splitter (PBS) and enters the arms $A$ and $B$. After a reflection in the arms the pulse re-enters the loop and is split into the four available modes, depending on the arm’s polarization rotation. For detection of the pulses, we place another weakly reflecting beam splitter ($R \approx 2\%$) in the loop and use a pair of PBSs and four superconducting nanowire single-photon detectors to discriminate the four internal states. By using single mode optical fibers of different lengths (328 and 338 m) in the arms $A$ and $B$, we can introduce a well-defined time delay of $\tau_{\text{pos}} = 95\text{ ns}$ between pulses that took different arms. By choosing the time delay to be longer than the pulse widths ($\approx 100\text{ ps}$) and detector dead times ($\approx 90\text{ ns}$), we can resolve the outcoupled pulses with different delays and associate them to unique time-bins (the recovery of detection efficiency is $> 95\%$ by that time). In order to achieve a good signal-to-noise ratio for high step numbers we perform measurements with two different initial power levels, which are then concatenated. This concatenation of two data sets is necessary since for a low power input the signal becomes too small after nine steps, while the high input powers cause detector saturation for the early steps and make a reliable probability extraction impossible. In each case we normalise the total intensity per step to one which is then equivalent to the walker’s probability distribution.

To understand the dynamics of the interferometer, it is instructive to follow what happens to a pulses com-
FIG. 2. Schemes for the Mach–Zehnder-type (a) and for the Michelson-type loop geometry (b). In both cases the walker is coupled into the loop via a partially reflective beam splitter and is polarization rotated in the loop by optical elements realising the operator $C_L$. At the polarizing beam splitter (PBS) the pulse is split according to its polarization and the two arms $A$ and $B$ with different lengths $L_A$ and $L_B$ introduce a well-defined time-delay between the constituents. In the Mach–Zehnder interferometer the pulses travel in the counter-clockwise direction, while in the Michelson geometry, both clockwise (denoted as $c$) and counter-clockwise ($cc$) travelling directions are used. The optical elements placed in the arms of the Michelson-type geometry realise the rotations $C_A$ and $C_B$ after the double passage. As a consequence, the walker can be characterised by four internal states (the two polarizations and two travelling directions) and its state can generally be described by Eq. (1). Graph representations of the 1D quantum walks by the Mach–Zehnder and Michelson-type geometries, respectively, are shown in (c) and (d), illustrating the role of the different polarization and propagation modes.

Particularly simple dynamics can be observed if the optical elements in the arms $A$ and $B$ are set up such that the net effect of the double passage and reflection is a rotation of the pulse polarization by $90^\circ$. In this case an initially counter-clockwise travelling pulse continues to travel in the counter-clockwise direction after returning from the arms $A$ and $B$. Therefore, the only role of the arms $A$ and $B$ is to provide a polarization dependent delay, while mixing of polarizations depends solely on the elements located inside the loop. By controlling the elements inside the loop, a wide range of general 1D quantum walk dynamics is accessible — limited only by the capabilities of the available optical components. In fact, the dynamics closely resembles that of previous time-multiplexing implementations of 1D quantum walks [18, 31], based on a Mach–Zehnder-type geometry (see Fig. 2 for comparison).

III. MATHEMATICAL DESCRIPTION

A. Time evolution of pulses as a quantum walk

For the purposes of mathematical description, we use a formal mapping between a wave mechanical superposition of spatially or temporally separated optical pulses and a quantum mechanical superposition of states of a photon [49] representing the quantum walker, as employed in our previous works [18, 19, 31, 41].

The state of a discrete-time quantum walker is described by $|\Psi\rangle$, a vector in the corresponding tensor product Hilbert space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_x$. For a DTQW on a line, the position Hilbert space $\mathcal{H}_x$ equals $l^2(\mathbb{Z})$, spanning all possible positions $x$ associated with the basis vectors $\{|x\rangle | x \in \mathbb{Z}\}$. The coin Hilbert space, $\mathcal{H}_c$, describes the internal degree of freedom. For a 1D walk a two-dimensional coin space is usually assumed, which facilitated the use of polarization for this purpose by a number of research groups (see e.g. [17, 18, 20, 42]). In a Michelson geometry (Fig. 2 (b)) the walker can addition-
ally be in a superposition of the two travelling directions in the loop, resulting in a four-dimensional coin space for a 1D walk. As described above we name the four basis states of \( \mathcal{H} \) as \{\( |cH\rangle, |cV\rangle, |ccH\rangle, |ccV\rangle \}. Any state \( |\Psi\rangle \) of the walker can then be written as

\[
|\Psi\rangle = \sum_{x \in \mathbb{Z}} (\alpha_{cH,x} |cH\rangle \otimes |x\rangle + \alpha_{cV,x} |cV\rangle \otimes |x\rangle + \alpha_{ccH,x} |ccH\rangle \otimes |x\rangle + \alpha_{ccV,x} |ccV\rangle \otimes |x\rangle)
\]

with the complex coefficients \( \alpha_{d,x} \) obeying \( \sum_d \sum_x |\alpha_{d,x}|^2 = 1 \).

The unitary evolution of a DTQW is determined by the coin operator \( \hat{C} \) acting on the internal degree of freedom, followed by the step operator \( \hat{S} \), which performs a conditional shift in the position \( x \); together we write \( |\Psi_{t+1}\rangle = \hat{S}\hat{C}|\Psi_t\rangle \). In the convention defined by the experimental setup (Fig. 2 (b)), \( \hat{S} \) shifts the basis states \( |cH\rangle \) and \( |ccV\rangle \) \((|ccH\rangle \) and \( |cV\rangle \)) one position to the right (left), which corresponds to earlier (later) arrival times, and simultaneously reverses the traveling direction; in quantum walk terminology such conditional shift combined with a reverse in direction is commonly referred to as a flip-flop step operator. Formally, the operator can be expressed as

\[
\hat{S} = \sum_x \left( |ccH\rangle \langle cH| \otimes |x - 1\rangle \langle x| + \langle cV| \langle ccV| \otimes |x + 1\rangle \langle x| \right).
\]

Since the position space is still one-dimensional but two different coin states indicate a step to the left and two to the right, the structure of the walk can be visualised as a line graph with doubled edges as illustrated in Fig. 2 (d).

The coin matrix describes the combined action of three \( 2 \times 2 \) polarization rotations defined by the three operations \( C_L \), \( C_A \) and \( C_B \) in the loop and the two arms, respectively. Note that the elements in the arms \( A \) and \( B \) are passed twice by each pulse entering the respective arm; by \( C_A \) and \( C_B \) we describe the full rotation accumulated by the time it re-enters the loop. To realize the desired polarisation rotations, we use quarter-wave plates (QWPs), half-wave plates (HWPs) and EOMs. In the polarization basis \{\( |H\rangle, |V\rangle \} \) the wave plates aligned at an angle \( \alpha \) are characterised by the matrices

\[
C_{\text{QWP}}(\alpha) = \frac{-i}{\sqrt{2}} \begin{pmatrix}
\cos 2\alpha + i \\
\sin 2\alpha
\end{pmatrix}
\]

\[
C_{\text{HWP}}(\alpha) = \begin{pmatrix}
\cos 2\alpha & \sin 2\alpha \\
\sin 2\alpha & -\cos 2\alpha
\end{pmatrix}
\]

respectively. The EOMs are aligned such that they are described by matrices

\[
C_{\text{EOM}}(\varphi) = \begin{pmatrix}
\cos \varphi & -i \sin \varphi \\
-i \sin \varphi & \cos \varphi
\end{pmatrix}
\]

with the phase \( \varphi \) depending on the voltage applied to the particular EOM during a particular time bin [31]. In all cases involving dynamic EOM switches we always align the quarter- and half-wave plates at \( \alpha = 45^\circ \), so that the matrices (3) and (4) commute with (5) and it is inconsequential in which order a pulse encounters them.

Since the elements in the loop do not mix counter propagating pulses, their effect can be described in the basis \{\( |cH\rangle, |cV\rangle, |ccH\rangle, |ccV\rangle \} \) by the block diagonal matrix

\[
C_{LL} = \begin{pmatrix}
C_{L,HH} & C_{L,HV} & 0 & 0 \\
C_{L,VH} & C_{L,VV} & 0 & 0 \\
0 & 0 & C_{L,HH} & C_{L,HV} \\
0 & 0 & C_{L,VH} & C_{L,VV}
\end{pmatrix}
\]

Due to the action of the PBS, the optical elements in the arms \( A \) and \( B \) mix pulses from different travelling directions, so that the total operation corresponds to the matrix

\[
C_{AB} = \begin{pmatrix}
C_{A,HH} & 0 & 0 & C_{A,HV} \\
0 & C_{B,VV} & C_{B,VH} & 0 \\
0 & C_{B,HV} & C_{B,HH} & 0 \\
C_{A,VH} & 0 & 0 & C_{A,VV}
\end{pmatrix}
\]

transforming e.g. \( |cH\rangle \) into a superposition of \( |cH\rangle \) and \( |ccV\rangle \). The coin matrix of the quantum walk arises as the product of these two matrices

\[
C = C_{AB}C_{LL}.
\]

When the polarization rotations are static in time, we can express the full coin operator as \( \hat{C} = C \otimes I_x \). However, due to the unique relation between time bins and positions and step number of the walker, we can program specific phase shifts \( \varphi_{l,x,A}, \varphi_{l,x,B} \) and \( \varphi_{l,x,L} \) to be realized for each position \( x \) by the three EOMs, thus making the coin operator position and time dependent, formulated as \( \hat{C}_l = \sum_x C_{l,x} \otimes |x\rangle \langle x| \). In this work we will only make use of the position dependence of the coin assuming temporally static operations.

**B. Analysis of the accessible coins**

The product form (8) does not cover all of \( U(4) \): a general matrix needs to satisfy additional criteria to allow such decomposition and thus be used as a coin matrix in our formalism. Namely, a given

\[
C = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix}
\]

\( C \in U(4) \), can be written in this form if and only if the vectors \( (c_{11}, c_{21}, c_{33}, c_{44}) \) and \( (c_{12}, c_{24}, c_{34}, c_{42}) \) as well as \( (c_{13}, c_{21}, c_{31}, c_{43}) \) and \( (c_{14}, c_{22}, c_{32}, c_{44}) \) are pairwise linearly dependent. The proof of this statement, included in the supplemental material S1, is constructive in the
sense that if there exists a decomposition Eq. (8) for a particular coin $C$, it tells us what transforms to use as $C_A$, $C_B$ and $C_L$.

A larger class of coins can be covered when the restriction is lifted that $C_{LL}$ acts the same on $c$ and $cc$-propagating pulses. This can be achieved e.g. by altering our setup such that counter-propagating pulses reach the loop EOM with a sufficient time difference, allowing the programming of different rotations. All of our experimental results are obtained with coin operators belonging to the first category of coins of the form Eq. (6), proving that it is far from limiting.

The full $U(4)$ can be recovered by employing a multi-step protocol $[33, 43, 44]$ using three steps. We base our protocol on the fact that any four-dimensional unitary matrix can be written as a product of two matrices of the form (8), which we prove rigorously in the supplemental material S2. Let us first consider a static coin operator of the form $C = C \otimes 1_L$. The application of $C$ on the pulses during each roundtrip is inseparably connected with applying the relative delay between the arms, expressed by the step operator (2), but we can leverage on the flip-flop nature of $\hat{S}$, namely that $\hat{S} \cdot \hat{S} = 1$. If we consider three consecutive roundtrips, in which the middle coin is an identity operation (achieved by setting $C_A = C_B = C_L = 1$), the overall action on the state becomes

$$|\Psi_{t+3}\rangle = \hat{S} \hat{C}_2 \hat{S} \hat{C}_1 |\Psi_t\rangle = \hat{S} \hat{C}_2 \hat{C}_1 |\Psi_t\rangle,$$

(10)
giving us the desired opportunity to act on $\hat{C}_1 |\Psi_t\rangle$ with another coin operation directly. Moreover, the transition from $|\Psi_t\rangle$ to $|\Psi_{t+3}\rangle$ has the form of a single step of a new quantum walk with the coin operator $\hat{C}_2 \hat{C}_1$, which has been proved universal. Finally, addressing the pulses in each time bin separately, Eq. (10) naturally extends to position- and time-dependent coin operators.

### IV. EXPERIMENTAL RESULTS

We present in this section the experimental results achieved by the setup realizing four-dimensional coin operators. The extra coin dimensions allowed us to conveniently realize cyclic graphs of various topologies.

To test the coherence properties of the setup we first perform measurements involving static waveplates only and no dynamic modulators. In a standard 1D Hadamard walk we find good agreement with the numeric model over 25 steps. Additionally, coherence involving all four internal states is measured by coupling the $c$ and the $cc$ directions statically in one arm. This is then followed by measurements of quantum walks on circles of different sizes using dynamic modulation of the coins, with mixing and non-mixing dynamics and varying input positions and polarizations. Exploring graphs beyond circles, we then demonstrate for the first time a quantum walk on more complex graph structures embedded in 2D containing nodes of non-uniform rank. We have chosen an illustrative example the figure eight graph, which includes a node that is connected by four edges, allowing us to test the dynamics by a genuine $4 \times 4$ coin operator permitted by Eq. (8).

#### A. Hadamard walk

Before realising complex protocols involving the full four-dimensional coin space we performed a coherence test of the setup by implementing a simple 1D walk constrained to only one travelling direction. To do so, we set $C_L$ to a Hadamard operation, realised up to a global phase by a HWP at $\alpha = 22.5^\circ$, yielding

$$C_{LL} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$  

(11)

The polarization rotations $C_A$ and $C_B$ in the arms are set to a polarization swap by introducing QWPs at the angle $\alpha = 45^\circ$ which are passed twice. Thus, up to a global $-i$ phase, the corresponding coin operator looks as

$$C_{AB} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$  

(12)

This maps $|ccH\rangle$ to $|cV\rangle$ and $|ccV\rangle$ to $|cH\rangle$, and the subsequent step operator (2) brings the direction back to
cc. Due to the absence of mixing of travelling directions, the pulses only ever travel in the loop in the counterclockwise direction, in which the walk was initiated.

The system reduces to a quantum walk with a two-dimensional coin, which can be described by the effective step and coin operators as

$$
\bar{S}_2 = \sum_{x \in \mathbb{Z}} \left( |R\rangle \langle R| \otimes |x + 1\rangle \langle x| + |L\rangle \langle L| \otimes |x - 1\rangle \langle x| \right)
$$

$$
C_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
$$

where we use $|R\rangle$ and $|L\rangle$ to follow the conventional notation for the up and down shifted components, respectively. The abstract states $|R\rangle$ and $|L\rangle$ correspond in the experiment to $|ccH\rangle$ and $|ccV\rangle$.

In this way we measure a standard Hadamard walk over 25 steps in near-perfect agreement with the theoretical expectation (Fig. 3), which confirms the high degree of coherence on the setup. As a figure of merit we use the polarisation resolved similarity between experimental and numerical probabilities defined as

$$
\mathcal{S} = \left| \sum_{d,x} \sqrt{P^{(\text{exp})}_{d,x} P^{(\text{num})}_{d,x}} \right|^2,
$$

for the relevant positions $x$ and the coin states $d$. In the figure captions we indicate this quantity averaged over the specified roundtrips.

### B. Coupled walks

In the next step we test the coherence properties when statically coupling both travelling directions. One scheme is presented in Fig. 4. Here, the QWP realising $C_A$ preserves the travelling direction by swapping the polarizations as explained earlier, while in the other arm $C_B$ is implemented by a QWP at $0^\circ$ and thus reverses the travelling directions in the subsequent step operation (Eq. (2)). This results in the coin operation

$$
C_{AB} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.
$$

Note that the loop coin $C_{LL}$ has no effect on the direction. In the loop a Hadamard operation is applied using a HWP at 22.5° as in Eq. (11). Again the strong similarity of 93.1% between simulation and experiment guarantees coherence even when both travelling directions are involved.

### C. Quantum walks on circles

After having confirmed the large degree of coherence provided by the Michelson loop we focus on harnessing the possibilities of the new geometry. To show the advantage of the Michelson setup we realise quantum walks on circles of different sizes, which involves position dependent coins. These are implemented by three fast-switching EOMs with the polarization rotation given in Eq. (5). The EOMs are placed in the loop and the two arms in addition to the static waveplates. In Fig. 5 we demonstrate how a circle can be formed in the graph of Fig. 2d by choosing two end points (here: $x = \pm 2$), allowing no coupling between c and cc components in the inner positions and no coupling from the end positions outwards. This leaves an effective walk on a circle of $2N$ sites if the two endpoints are $N$ positions apart. In the following we label the sites using a coordinate $m = 0$ through $m = 2N - 1$. We can describe this walk using a two-dimensional coin and a step operator as in Eq. (13), but with an additional periodic boundary condition $|m\rangle \equiv |m + 2N\rangle$. We have measured the results of applying both mixing or non-mixing operations on the circle. Note that instead of the conventional Hadamard operation as given in Eq. (13) we use here another balanced matrix with different complex phases,

$$
H' = C_{\text{EOM}(45^\circ)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.
$$
FIG. 6. Walk on a circle with 16 nodes, i.e. with boundaries at \(x = \pm 4\) (non-mixing operation on the circle, \(|ccD\rangle\) input), displayed separately for clockwise and counter-clockwise propagating components in numerics and experiment. One can immediately see the jump from \(cc\) to \(c\) (and vice versa) at \(x = \pm 4\) in step 4 to 5 (step 11 to 12). Since all the outer positions are unoccupied up to small switching inaccuracies, we will restrict the plot range to the relevant positions numbered as sites on the circle \(m = 0, \ldots, 15\) from now on (as exemplary indicated in Fig. 5 for a smaller circle). The polarisation resolved similarity averaged over 19 steps is 87.1 %.

This is because (13) cannot be directly realised by an EOM, which we need for the position dependence. Because this gives the same 50:50 splitting, we refer to (16) as Hadamard-like coin. For the different settings and the associated physical implementation see table I. In Fig. 6 we show the experimental and numerical data for a walk taking place on a circle with 16 nodes, i.e. boundaries at \(x = \pm 4\) in the original linear graph.

For this illustrative example we realise a non-mixing coin on the circle. To this end, we need to employ all three EOMs for the realisation of the dynamic switchings discriminating between inner and boundary positions of the graph. The input polarization \(|D\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\) starting in the \(cc\) direction initiates two counter-propagating peaks in the walker’s wave function in an equal superposition, regularly meeting in \(x = 0\). In Fig. 7 we plot a similar 8-node circle for both the non-mixing and the \(H'\) operation. We plot only over the relevant positions \(m = 0, \ldots, 7\) and find a high agreement between experiment and numerics.

An interesting effect in our experiment is the equidistribution of the walk, by which we mean that the probability distribution corresponding to the wave function becomes close to uniform. The effect can be observed for example in Fig. 7c. Although equidistribution is not generally exhibited by QWs, a quantum walk on a circle of 8 nodes started from a localized state reaches a uni-

FIG. 7. (b) Free passage walk with a similarity \(S = 85.5\%\) and (c) Hadamard-like \(H'\) walk with \(S = 78.8\%\) (both averaged over 19 steps) on an 8-site circle (a) with \(|ccD\rangle\) input.
FIG. 8. Equidistribution for a QW on an 8-site circle as analysed in Fig. 7. Upper Panel: Intensity distribution in step 11 of the experimental (red) and numerical (blue) data for almost ideal mixing (polarization is traced out). Lower panel: Similarity (see Eq. (14)) traced over polarization and only for the relevant position 1, 3, 5, 7 of the experimental (red dots) and numerical (blue dots) data and the flat distribution over the 4 occupied positions versus the roundtrip number. In both cases the deviation of the experimental data from the numeric can be explained through imperfect switchings at the boundaries such that a small part of the intensity leaves the circle sites. Since we present the data without renormalisation over the circle sites only, but take the “lost” intensity into account, the walker’s overall intensity over the circle positions only is less than 1.

FIG. 9. Hadamard walk on an 16-site circle with $|\text{ccD}\rangle$ input. The polarisation resolved similarity averaged over 19 steps is 89.1 %.

FIG. 10. (a) Configuration of an asymmetric circle walk on 10 sites. The walk is started at $x = 0$ (orange arrow), which corresponds to position $m = 2$ on the circle and in the chessboard plot axes. The input state is $|\text{ccD}\rangle$ and (b) shows the step evolution of a non-mixing walk (similarity averaged over 19 steps: 88.9 %), while (c) displays the evolution of an effective Hadamard walk in such a configuration (similarity: 79.9 %).

form distribution in steps 10 through 14. Note that even (odd) positions are unreachable in odd (even) step numbers with a localised initial state, it is understood that the distribution is uniform over the set of positions that are allowed by the dynamics. In an earlier work this property has been studied in QWs on the line [50], where the term mixing was used. We, however, find it more appropriate to reserve the use of the term mixing for a property that arises as a time average [51–53], acknowledging that
### (a) Non-mixing

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<th>arms</th>
<th>loop</th>
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<td>$C_{HWP}(45^\circ)$</td>
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<tr>
<td>EOM</td>
<td>$C_{EOM}(0^\circ)$</td>
<td>$C_{EOM}(0^\circ)$</td>
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<tr>
<td>resulting action</td>
<td>$C_A = C_B = -iX$</td>
<td>$C_L = X$</td>
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<tr>
<td>EOM</td>
<td>$C_{EOM}(-90^\circ)$</td>
<td>$C_{EOM}(-90^\circ)$</td>
</tr>
<tr>
<td>resulting action</td>
<td>$C_A = C_B = 1$</td>
<td>$C_L = i1$</td>
</tr>
</tbody>
</table>

### (b) Hadamard-like

<table>
<thead>
<tr>
<th>inner positions</th>
<th>arms</th>
<th>loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>static elements</td>
<td>$[C_{QWP}(45^\circ)]^2$</td>
<td>$C_{QWP}(45^\circ)$</td>
</tr>
<tr>
<td>EOM</td>
<td>$C_{EOM}(0^\circ)$</td>
<td>$C_{EOM}(0^\circ)$</td>
</tr>
<tr>
<td>resulting action</td>
<td>$C_A = C_B = -iX$</td>
<td>$C_L = H'$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>end positions</th>
<th>arms</th>
<th>loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>static elements</td>
<td>$[C_{QWP}(45^\circ)]^2$</td>
<td>$C_{QWP}(45^\circ)$</td>
</tr>
<tr>
<td>EOM</td>
<td>$C_{EOM}(-45^\circ)$</td>
<td>$C_{EOM}(-45^\circ)$</td>
</tr>
<tr>
<td>resulting action</td>
<td>$C_A = C_B = H'$</td>
<td>$C_L = 1$</td>
</tr>
</tbody>
</table>

**TABLE I.** Experimental realisation of the coin settings for QWs on circles for non-mixing and Hadamard-like operation. The static elements act the same way in every position, while the dynamic EOM can perform distinct operations for the inner and end positions. The total action can be computed by taking the products of static WP and EOM matrices given in Eqs. (3)–(5). Each WP needs to be considered once in the loop, and twice in the arms due to the reflection, giving rise to the square of the operators. Note that the EOM is only switched on for one direction (when the pulses pass it after the reflection at the mirror) in order to keep the number of overall switches low. The resulting operations are indicated with identity $I$, the Pauli $X$ gate and $H'$ as given in Eq. (16).

unitary processes generally do not converge to a stationary distribution. We analyse the equidistribution in detail in Fig. 8, where we present the intensity histogram for roundtrip 11 in which the experimental data shows nearly equal intensity at all four occupied positions. In the second panel we track the evolution of the walker’s probability distribution over the roundtrip numbers by comparing it to the uniform distribution using the similarity (see Eq. (14)). We can extract an equidistribution time of approximately 10–12 roundtrips in agreement with the numerical model. This equidistribution effect is likely linked to the perfect state revival [54]: an initially localized state goes through a uniform distribution at half of the period, along with some neighbouring steps. An example of a smaller circle with 4 sites showing the revival of the initial state in several repetitions is presented in Fig. S1 in the supplemental material.

By reprogramming the switching times of the three EOMs we can easily realise larger circles (see Fig. 9 for 16 nodes and a Hadamard-like walk), or a circle with an asymmetric incoupling position (see Fig. 10), where the asymmetry is only associated to the graph picture as all positions on the circle are naturally equivalent. In all situations we can easily program the effective coin mixing around the circle according to table I and observe a high overlap of numeric and experimental data.

**FIG. 11.** (a) Modified ladder graph, equivalent to a walk on a figure eight, shown here for a graph with 15 nodes. Note the coupling at $x = 0$, which involves all four possible links. (b) Numerics and experimental data of the intensity evolution of a figure eight quantum walk with a non-mixing coin and $|ccD\rangle$ input. The polarisation resolved similarity averaged over 15 steps is 90.3 %. (c) same as (b) but with effective Hadamard splitting. (Similarity: 90.5 %)

D. Figure eight

Circles already represent a significant step beyond Euclidean lattices, offering grounds for studying phenomena like mixing and revivals. However, since each node of a circle is connected only to two neighbours, these graphs do not make use of the genuine four-dimensional coin op-
erator that our setup can provide. To study the implementation of a genuine four-dimensional coin operator, we realise a QW on a figure eight graph, which consists of two circles joined by a central node comprising four links to its neighbours (see Fig. 11). Dynamics on the nodes of the circle are experimentally implemented, analogously to the circles, by dynamically controlled elements as listed in Table I.

In order to add an additional link at position $x = 0$ (equivalently, $m = 7$) we perform in the non-mixing setting an extra switch with the EOMs in the arms by $-90^\circ$ compensating the polarization swap by the static elements $\sqrt{C_A} = \sqrt{C_A} = C_{QWP}(45^\circ)$, while the loop operation is given only by the passive HWP swapping the polarization $C_L = C_{HWP}(45^\circ)$, thus e.g. the mode $|cH\rangle$ is mapped to $|cV\rangle$ and vice versa. In the Hadamard-like setting the static mixing coin in the loop $C_L = C_{QWP}(45^\circ)$ is accompanied by the arm coin operations $H' = C_{QWP}^2(45^\circ) \cdot C_{EOM}(-45^\circ)$ employing again an additional EOM switch. In the latter case, the resulting $C_A = C_B = H'$ combines with the rotation $C_L = H'$ in the loop to form a full-rank four-dimensional coin matrix (see Eq. (8))

$$C = \frac{1}{2} \begin{pmatrix}
  1 & -i & 1 & i \\
  -i & 1 & 1 & i \\
  1 & i & 1 & -i \\
  i & 1 & -i & 1
\end{pmatrix}. \quad (17)$$

The results for both of the settings are presented in Fig. 11 (b) and (c), respectively. One can clearly observe the light reappearing at node 7 after one cycle around the right and the left half of the figure eight. Again, the coherence properties ensure a high agreement of experimental and numerical data even on such a complex graph structure.

V. CONCLUSION AND OUTLOOK

Fully exploiting the potential of discrete time quantum walks requires a reliable and comprehensive implementation of higher dimensional coin operators. To realize DTQW dynamics with genuine four-dimensional coin operators we have developed a novel experimental platform based on a looped Michelson interferometer. We have carried out several experiments with increasing complexity, demonstrating the accuracy, stability and capacity of our setup to realize four-dimensional coins. We started from a conventional Hadamard walk on the line, which shows high coherence over many steps, but essentially uses only a two-dimensional subspace of the available coin operations. Next we presented a system which uses the four-dimensional coin space in a nontrivial way, precluding a possible description as a walk with an effectively two-dimensional coin. Based on this we developed a scheme to realise quantum walks on circles with programmable sizes over many steps by dynamic coin operations. The four dimensions of the coin space allowed us to overcome experimentally costly multi-step schemes. Finally, we presented a QW on a non-trivial figure eight graph, which includes periodic boundary conditions at both sides and a central node with links to all its four neighbours. This structure can only be generated and controlled by a four-dimensional coin operator, which we used to present examples with a non-mixing and a Hadamard-like coin.

Our theoretical analysis yielded a constructive proof of how to implement a four-dimensional coin matrix by $2 \times 2$ polarization rotations and proving the universality of the four-dimensional coin with few and straightforwardly implementable modifications. In particular, we proved that two subsequent applications of the coin matrix (8), resulting from polarization transformations at three different places of the setup, are sufficient to reach any desired four-dimensional coin transformation. This simple three-step approach achieves universality without structural changes to the layout of the experiment. Thus any coin, such as the Grover and Fourier coins, are achievable, going far beyond the capacities of previous multi-step protocols based on two-dimensional coins [33, 43, 44]. The necessary step numbers can be achieved by adding a deterministic in and outcoupling instead of the partially reflecting mirrors and thus maximizing the roundtrip efficiency. Using this approach almost 40 steps were recently demonstrated in the Mach–Zehnder setup [41].

Our quantum walk with higher dimensional coin space has potential applications in the context of lazy walks, for which at least a three dimensional internal state is required [55–58], quantum game theory [59], and topological effects going beyond the split-step scheme [45]. The application of the higher dimensional coin to realize closed circles allows the study of magnetic walks [60], in which closed paths in the underlying graph geometry are essential. Quantum walks on non-trivial graph structures are also used in the framework of search [61] and graph isomorphism testing [62] involving distinguished nodes with several neighbours. Our figure eight graph, with the central node connected to four neighbours, is a minimal example of such a structure.

To use the Michelson—type geometry for realizing a full second lattice dimension we can consider extensions analogous to the 2D quantum walk on a Mach–Zehnder setup [19]. The universal coin operator would allow for the first time the experimental study of localization and trapping effects on 2D grids. Namely, genuine four-dimensional coins could have an effect analogous to the spin-orbit coupling [68] inducing an Anderson transition, an effect that could not be observed in split-step walks [67]. Additionally implementing periodic boundary conditions, both the trapping [48, 52, 63] and spreading of a quantum walk with a genuine four-dimensional Grover coin [47], as well as search protocols [6, 64, 65], and graph geometries [66] could be studied in our system.

These examples rely on four dimensional coin operators
providing a structure to the dynamics not attainable using lower dimensional coin space dynamics. Our platform provides the first instance of an extensible realisation of a quantum walk with four-dimensional coin operators with precise dynamic control, paving the way to experimental implementations of many important applications relying on genuine higher dimensional coins.

ACKNOWLEDGMENTS

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[35] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, and


Supplemental Material: A photonic quantum walk with a 4-dimensional coin

S1: COINS ACHIEVABLE IN ONE ROUND TRIP

Here we prove the following related to Sec. III in the main text:

**Theorem 1:** A unitary matrix

\[
C = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{21} & c_{22} & c_{23} & c_{24} \\
c_{31} & c_{32} & c_{33} & c_{34} \\
c_{41} & c_{42} & c_{43} & c_{44}
\end{pmatrix},
\]  

(S1)

can be written in the form (see Eq. (8))

\[
C = \begin{pmatrix}
a_{HH} & 0 & 0 & a_{HV} \\
0 & b_{VV} & b_{VH} & 0 \\
0 & b_{HV} & b_{HH} & 0 \\
a_{VH} & 0 & 0 & a_{VV}
\end{pmatrix} \cdot \begin{pmatrix}
l_{HH} & l_{HV} & 0 & 0 \\
l_{VH} & l_{VV} & 0 & 0 \\
0 & 0 & l_{HH} & l_{HV} \\
0 & 0 & l_{VH} & l_{VV}
\end{pmatrix},
\]  

(S2)

where

\[
\begin{pmatrix}
a_{HH} & a_{HV} \\
0 & a_{VV}
\end{pmatrix}, \quad \begin{pmatrix}
b_{HH} & b_{HV} \\
0 & b_{VV}
\end{pmatrix}, \quad \begin{pmatrix}
l_{HH} & l_{HV} \\
l_{VH} & l_{VV}
\end{pmatrix}
\]  

are unitary matrices, if and only if the matrices

\[
\begin{pmatrix}
c_{11} & c_{12} \\
c_{21} & c_{22} \\
c_{31} & c_{32} \\
c_{41} & c_{42}
\end{pmatrix}, \quad \begin{pmatrix}
c_{13} & c_{14} \\
c_{23} & c_{24} \\
c_{33} & c_{34} \\
c_{43} & c_{44}
\end{pmatrix}
\]  

(S4)

both have rank one (i.e., linearly dependent rows or columns).

The implication from (S2) to the latter property follows trivially from performing the matrix multiplication, but it’s instructive to have an explicit expansion:

\[
\begin{pmatrix}
c_{11} & c_{12} \\
c_{41} & c_{42} \\
c_{23} & c_{24} \\
c_{33} & c_{34}
\end{pmatrix} = \begin{pmatrix}
a_{HH} l_{HH} & a_{HH} l_{HV} \\
a_{VV} l_{HH} & a_{VV} l_{HV} \\
b_{VV} l_{HH} & b_{VH} l_{HV} \\
b_{HH} l_{HH} & b_{HV} l_{HV}
\end{pmatrix}
\]  

(S5)

Let us now treat the opposite implication, i.e., assume that the two matrices in (S4) are of unit rank, so their elements must be of the form

\[
\begin{pmatrix}
c_{11} & c_{12} \\
c_{41} & c_{42} \\
c_{23} & c_{24} \\
c_{33} & c_{34}
\end{pmatrix} = \begin{pmatrix}
\alpha_1 \beta_1 & \alpha_2 \beta_1 \\
\alpha_1 \beta_2 & \alpha_2 \beta_2 \\
\alpha_1 \beta_3 & \alpha_2 \beta_3 \\
\alpha_1 \beta_4 & \alpha_2 \beta_4
\end{pmatrix},
\]  

\[
\begin{pmatrix}
c_{13} & c_{14} \\
c_{43} & c_{44} \\
c_{21} & c_{22} \\
c_{31} & c_{32}
\end{pmatrix} = \begin{pmatrix}
\gamma_1 \delta_1 & \gamma_2 \delta_1 \\
\gamma_1 \delta_2 & \gamma_2 \delta_2 \\
\gamma_1 \delta_3 & \gamma_2 \delta_3 \\
\gamma_1 \delta_4 & \gamma_2 \delta_4
\end{pmatrix}
\]  

(S6)
for some \( \alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C} \). Any matrix formed by such elements can be written in the following product form,

\[
C = \begin{pmatrix}
\beta_1 & 0 & 0 & \delta_1 \\
0 & \delta_3 & \beta_3 & 0 \\
0 & \delta_4 & \beta_4 & 0 \\
\beta_2 & 0 & 0 & \delta_2 \\
\end{pmatrix}
\begin{pmatrix}
\alpha_1 & \alpha_2 & 0 & 0 \\
0 & \gamma_1 & \gamma_2 & 0 \\
0 & 0 & \alpha_1 & \alpha_2 \\
0 & 0 & \gamma_1 & \gamma_2 \\
\end{pmatrix}.
\] (S7)

This is already close to the desired form (S2) but nothing so far guarantees that the two matrices forming the right-hand side are also unitary and thus realisable by separate physical transforms.

It is easy to show that if \( \alpha_1 \) and \( \alpha_2 \), or \( \gamma_1 \) and \( \gamma_2 \), were simultaneously zero, \( C \) would be singular. In all the other cases there is some freedom in decomposing the left-hand sides of (S6), so without loss of generality we can assume that \( |\alpha_1|^2 + |\alpha_2|^2 = |\gamma_1|^2 + |\gamma_2|^2 = 1 \).

The unitarity of \( C \) postulates that the norm of its first and last row must be equal to 1 and their scalar product must vanish. With the above assumption these equations take the forms

\[
|\beta_1|^2 + |\delta_1|^2 = 1, \\
|\beta_2|^2 + |\delta_2|^2 = 1, \\
\beta_1 \beta_2 + \delta_1 \delta_2 = 0.
\] (S8)

This simply says that the matrix

\[
\begin{pmatrix}
\beta_1 \\
\beta_2 \\
\end{pmatrix}
\] (S9)

is unitary. In (S7) these four elements take positions of the elements \( a_{ij} \) of (S2). Similarly from the middle two rows of (S7) we derive the unitarity of

\[
\begin{pmatrix}
\beta_3 \\
\beta_4 \\
\end{pmatrix}
\] (S10)

or \( (b_{ij}) \).

We also require that the first and second column of \( C \) are normalized and orthogonal vectors. Note that the unitarity of (S9) and (S10) also implies

\[
|\beta_1|^2 + |\beta_2|^2 = |\beta_3|^2 + |\beta_4|^2 = |\delta_1|^2 + |\delta_2|^2 = |\delta_3|^2 + |\delta_4|^2 = 1.
\] (S11)

Using the last equality, the row orthonormality condition gives the following equations,

\[
|\alpha_1|^2 + |\gamma_1|^2 = 1, \\
|\alpha_2|^2 + |\gamma_2|^2 = 1, \\
\alpha_1 \gamma_2 + \gamma_1 \alpha_2 = 0,
\] (S12)

which again are nothing else than the conditions on unitarity of

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 \\
\gamma_1 & \gamma_2 \\
\end{pmatrix}
\] (S13)

forming the blocks of the latter matrix in (S7).

In conclusion, the conditions stated by the theorem only allow matrices exactly of the form (S2) where the submatrices corresponding to \( C_A, C_B, C_L \) are all unitary. Their elements can easily be reconstructed using the following algorithm:

1. Build the matrices (S4) and take any decomposition of the form (S6), the existence of which is guaranteed by the assumptions.

2. Multiply the vectors \( (\alpha_1, \alpha_2) \) and \( (\gamma_1, \gamma_2) \) by some constants to achieve norms of one, dividing \( (\beta_j) \) and \( (\delta_j) \) by the same constants to keep products invariant.

3. Compare (S6) with (S5) to find correspondence between \( \alpha_j, \beta_j, \gamma_j, \delta_j \) and \( a_{ij}, b_{ij}, l_{ij} \).
Note: The same derivation can be repeated with minimal changes when the two diagonal blocks of the latter matrix of (S2) are not required to be equal, only unitary (as would correspond to transforming the clockwise- and counter-clockwise-propagating pulses in the loop independently). The two matrices in (S4) then need to be replaced by four matrices
\[
\begin{pmatrix}
C_{11} & C_{14} \\
C_{21} & C_{24}
\end{pmatrix}, \quad \begin{pmatrix}
C_{12} & C_{13} \\
C_{22} & C_{23}
\end{pmatrix}, \quad \begin{pmatrix}
C_{31} & C_{34} \\
C_{41} & C_{44}
\end{pmatrix}, \quad \begin{pmatrix}
C_{32} & C_{33} \\
C_{42} & C_{43}
\end{pmatrix}.
\] (S14)

**S2: ACHIEVING UNIVERSALITY**

As argued above, a possible universal recipe for achieving any coin is the following three-step protocol:

**Step 1:** apply a coin \(C_1\), evolve over one round trip,

**Step 2:** let the wave packets finish one full round trip with a trivial coin,

**Step 3:** apply another coin \(C_2\), finish the round trip.

This builds upon the flip-flop nature of the step operator: two applications thereof amount to the identity map. So if the coin is left trivial (\(C_A = C_B = C_L = 1\)) in Step 2, the application of \(\hat{S}\) in Step 2 negates any displacement made in Step 1 and returns the internal state to what it was immediately after the application of \(\hat{C}\) in Step 1. After Step 3 the internal state is thus effectively transformed by the product \(C_2C_1\) and subject to just one flip-flop displacement, according to the final coin state. The total action of these three round trips thus can be perceived as a single step of a quantum walk with a more general coin.

This coin becomes indeed completely general, if we allow that the \(C_{LL}\) block of Eq. (11) can be controlled separately for the c and cc polarizations (upper-left and lower-right blocks):

**Theorem 2:** Let \(C\) be a generic \(U(4)\) matrix. Then two transforms of the form
\[
C_t = \begin{pmatrix}
a_{HH,t} & 0 & 0 & a_{HV,t} \\
0 & b_{VV,t} & b_{HV,t} & 0 \\
0 & b_{HV,t} & b_{HH,t} & 0 \\
a_{VH,t} & 0 & 0 & a_{V,V,t}
\end{pmatrix},
\] (S15)
can be found, \(C_1, C_2\), such that \(C = C_2C_1\), with the individual submatrices \(a, b, l, l'\) in \(U(2)\). Moreover, up to a global phase correction factor, the four submatrices can be sought in \(SU(2)\).

We will prove this theorem constructively, using bra-ket notation on \(\mathbb{C}^2\): let in this section ket denote a two-element column vector and a bra with the same symbol its conjugate, a row vector composed of complex conjugate elements. Namely, we pair the unknowns of the decomposition in the following objects:
\[
|p\rangle := \begin{pmatrix} l_{HH,2} \\ l_{HV,2} \end{pmatrix}, \quad |q\rangle := \begin{pmatrix} l_{HV,2} \\ l_{V,V,2} \end{pmatrix},
\]
\[
|r\rangle := \begin{pmatrix} l'_{HH,2} \\ l'_{HV,2} \end{pmatrix}, \quad |s\rangle := \begin{pmatrix} l'_{HV,2} \\ l'_{V,V,2} \end{pmatrix},
\] (S16)

\((P) := (l_{HH,1} \ l_{HV,1})
\), \((Q) := (l_{V,H,1} \ l_{V,V,1})
\), \((R) := (l'_{HH,1} \ l'_{HV,1})
\), \((S) := (l'_{V,H,1} \ l'_{V,V,1})\).

The condition on unitarity of \(C_{LL}\) then translates into the requirement that \((|p\rangle, |q\rangle), (|r\rangle, |s\rangle), (|P\rangle, |Q\rangle), \) and \((|R\rangle, |S\rangle)\) are four (not necessarily different) orthonormal bases.

We will show that the decomposition stated by Theorem 2 exists even with a further restriction
\[
a_{ij,2} = b_{ij,2} = \delta_{ij},
\] (S17)
i.e., the \(C_A, C_B\) matrices in Step 2 being trivial. In the following \(a_{ij}\) and \(b_{ij}\) will thus denote \(a_{ij,1}, b_{ij,1}\) for brevity.
If we split the required coin matrix $C$ into $2 \times 2$ blocks as

$$C = \begin{pmatrix} C_{TL} & C_{TR} \\ C_{BL} & C_{BR} \end{pmatrix}, \quad (S18)$$

the equation

$$C = C_2C_1 \ldots a_{HH} < 1.$$  

Then both $a_{VH}$ and $a_{HV}$ are nonzero and $|s\rangle$ and $|S\rangle$ satisfy

$$|s\rangle = \frac{1}{a_{VH}} C_{BL} |P\rangle, \quad |S\rangle = \frac{1}{a_{HV}} C_{TR}^\dagger |p\rangle. \quad (S26)$$

We are also given the unitarity conditions of $C$:

$$C^\dagger C = 1, \quad CC^\dagger = 1. \quad (S21)$$

In the block form (S18), the former becomes

$$\begin{pmatrix} C_{TL}C_{TL} + C_{BL}^{\dagger}C_{BL} & C_{TL}C_{TR} + C_{BL}^{\dagger}C_{BR} \\ C_{TR}^\dagger C_{TL} + C_{BR}^{\dagger}C_{BL} & C_{TR}^\dagger C_{TR} + C_{BR}^{\dagger}C_{BR} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (S22)$$

and the latter

$$\begin{pmatrix} C_{TL}C_{TL}^\dagger + C_{TR}C_{TR}^\dagger & C_{TL}C_{BL}^\dagger + C_{TR}C_{BR}^\dagger \\ C_{BL}C_{TL}^\dagger + C_{BR}C_{TR}^\dagger & C_{BL}C_{BL}^\dagger + C_{BR}C_{BR}^\dagger \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (S23)$$

Plugging in (S20), we find that if such decomposition exists, it must satisfy

$$|a_{HH}|^2 = |a_{VH}|^2 = 1 - |a_{HH}|^2 = 1 - |a_{VH}|^2,$$

$$|b_{HH}|^2 = |b_{HV}|^2 = 1 - |b_{HH}|^2 = 1 - |b_{HV}|^2. \quad (S24)$$

Along with the orthonormality of $(|p\rangle , |q\rangle)$ etc., the equations (S20) strongly resemble singular value decompositions (SVDs): indeed, they would become SVDs of the left-hand side matrices if, furthermore, the $a_{ij}$ and $b_{ij}$ coefficients were real and nonnegative. Without loss of generality, we can thus postulate that the first line is the actual SVD, i.e., that $|p\rangle$ and $|q\rangle$ are left-singular vectors, $|P\rangle$ and $|Q\rangle$ right-singular vectors and $a_{HH}$ and $b_{VH}$ the singular values of $C_{TL}$, and see if we can satisfy the other three lines with this choice.

Given $|P\rangle$ and $|Q\rangle$, we can apply both sides of the third line of (S20) on them, obtaining

$$C_{BL} |P\rangle = a_{VH} |s\rangle, \quad C_{BL} |Q\rangle = b_{HV} |r\rangle. \quad (S25)$$

If the magnitude of at least one of the coefficients $a_{VH}$ or $b_{HV}$ is known to be nonzero (that is, per (S24), unless the singular values of $C_{TL}$ were both 1), the corresponding $|s\rangle$ or $|r\rangle$ is determined up to a complex phase. If both are, they are guaranteed to be orthonormal by (S23) and (S24). If $a_{VH}$ or $b_{VH}$ is zero, we complement $|s\rangle$ as an orthonormal partner of $|r\rangle$ or vice versa, respectively, with an arbitrary phase. In either case, the choice of the phase of the two vectors leaves $a_{VH}$ and $b_{HV}$ completely determined. The case $a_{VH} = b_{HV} = 0$ will be handled separately near the end of the proof.

Taking the Hermitian conjugate of the second equation of (S20) we find the vectors $|r\rangle$ and $|s\rangle$ and the numbers $a_{HV}$, $b_{VH}$ in a complete analogy to the above, leaving the same exceptional case.

After these steps, the last equation does not contain any undetermined vectors, so we need to prove that it is not a contradiction.

Assume $a_{HH} < 1$. Then both $a_{VH}$ and $a_{HV}$ are nonzero and $|s\rangle$ and $|S\rangle$ satisfy

$$|s\rangle = \frac{1}{a_{VH}} C_{BL} |P\rangle, \quad |S\rangle = \frac{1}{a_{HV}} C_{TR}^\dagger |p\rangle. \quad (S26)$$
We can then study
\[ C_{BR} |S\rangle = \frac{1}{a_{HV}} C_{BR} C_{TR}^\dagger |p\rangle = -\frac{1}{a_{HV}} C_{BL} C_{TL}^\dagger |p\rangle = -\frac{a_{HH}^*}{a_{HV}} C_{BL} |P\rangle = -a_{HH}^* \frac{a_{HV}}{a_{HV}} |s\rangle, \tag{S27} \]
where from step to step we used (S26), (S23) (lower-left block), (S20) (first line), and (S20) (third line). This shows that \( C_{BR} \) indeed maps \(|s\rangle\) to a multiple of \(|s\rangle\), as (S20) requires, but also gives a concrete value to \( a_{VV} \) and shows, along with (S24), that \( a_{ij} \) together form a \( U(2) \) matrix.

If \( a_{HH} = 1 \) (but \( b_{VV} < 1 \)), we can’t use (S26), but we have
\[ |r\rangle = \frac{1}{b_{HV}} C_{BL} |Q\rangle, \quad |R\rangle = \frac{1}{b_{VH}^*} C_{TR}^\dagger |q\rangle. \tag{S28} \]
We can still prove that \( C_{BR} \) acting on \(|S\rangle\) produces some vector orthogonal to \(|r\rangle\), which in turn is a multiple of \(|s\rangle\): for this, we consider
\[ \langle r| C_{BR} |S\rangle = \frac{1}{b_{HV}} \langle Q| C_{BL}^\dagger C_{BR} |S\rangle = - \frac{1}{b_{HV}} \langle Q| C_{TL}^\dagger C_{TR} |S\rangle = 0. \tag{S29} \]
The last step follows from the fact that for \( a_{HH} = 1, a_{HV} = 0 \) and thus \( C_{TR} |S\rangle = 0 \). We don’t learn the phase of \( a_{VV} \), as it depends on the arbitrary phases of both \(|S\rangle\) and \(|s\rangle\), but with \( a_{HV} = a_{VV} = 0 \) the \( a_{ij} \) matrix is unitary for any choice.

The emergence of the last term of the last equation of (S20) and the value of \( b_{HH} \) are handled similarly, resulting in \( b_{ij} \) being unitary and the system (S20) being consistent with our solution. Since the properties of \(|p\rangle, |q\rangle, \ldots, |R\rangle, |S\rangle\) also guarantee unitarity of \( l_{ij,1}, l_{ij,2}^*, l_{ij,2}, \) and \( l_{ij,2} \), this completes the decomposition.

We left out only one special case, \( a_{HH} = b_{VV} = 1 \). But this case is trivial: now \( C_{TL} \) is of the form
\[ C_{TL} = |p\rangle \langle P| + |q\rangle \langle Q| \tag{S30} \]
and so is unitary, \( C_{TR} \) and \( C_{BL} \) are zero, and \( C_{BR} \) is unitary again. Matrices of this block form can be realised in a single round trip; if necessary, a three-step protocol can be made trivially by taking \( C_1 = C, C_2 = 1 \). This last remaining case finishes the main part of the proof.

Restricting the \( a, b, l, l' \) submatrices to be special unitary is easy by the degrees of freedom encountered throughout the construction above. We will first consider the generic case where the off-diagonal blocks of \( C \) are nonzero.

We can follow closely the same algorithm as above but in the beginning, instead of using the SVD of \( C_{TL} \) directly, we fix the phases of the basis vectors so that the matrices \( l_{ij,2} = \langle |p\rangle |q\rangle \rangle \) and \( l'_{ij,1} = \langle |R\rangle |S\rangle \rangle \) become unimodular. This in general changes the complex phases of \( a_{HV} \) and of \( b_{HV} \). In the next steps we also choose the new base pairs so that they form matrices of determinant 1.

This only leaves the choice of balancing the phase between the two vectors in each of the four pairs. For example, multiplying \(|p\rangle\) by \( e^{i\varphi} \) and \(|q\rangle\) by \( e^{-i\varphi} \) leaves \( (l_{ij,2}) \) unimodular and becomes a no-operation if compensated by simultaneously multiplying \( a_{HH} \) and \( a_{HV} \) by \( e^{-i\varphi} \) and \( b_{VH} \) and \( b_{VV} \) by \( e^{i\varphi} \). But this amounts to a phase change in one row of the matrix \( a_{ij} \) and the opposite phase change in one row of \( b_{ij} \). In such a transform, all the matrices keep their determinants except the latter two, whose determinants are modified by mutually opposite phases. At a certain phase these become the same, and the common phase of the two matrices can be factored out of the decomposition as a unphysical complex prefactor.

In the special case
\[ C = \begin{pmatrix} C_{TL} & 0 \\ 0 & C_{BR} \end{pmatrix}, \tag{S31} \]
we find angles \( \alpha, \beta \) such that
\[ \det C_{TL} = e^{2i(\alpha + \beta)}, \quad \det C_{BR} = e^{2i(\alpha - \beta)}. \tag{S32} \]
\[ C = e^{i\alpha} \text{diag}\{e^{i\beta}, e^{i\beta}, e^{-i\beta}, e^{-i\beta}\} \begin{pmatrix} e^{-i\beta} C_{TL} & 0 \\ 0 & e^{i\beta} C_{BR} \end{pmatrix}, \] (S33)

which corresponds to choosing

\[
\begin{pmatrix}
  l_{HH,1} & l_{HV,1} \\
  l_{VH,1} & l_{VV,1}
\end{pmatrix} = e^{-i\alpha} C_{TL},
\]
\[
\begin{pmatrix}
  l'_{HH,1} & l'_{HV,1} \\
  l'_{VH,1} & l'_{VV,1}
\end{pmatrix} = e^{-i\beta} C_{BR},
\]
\[
\begin{pmatrix}
  a_{HH} & a_{HV} \\
  a_{VH} & a_{VV}
\end{pmatrix} = \text{diag}\{e^{i\beta}, e^{-i\beta}\},
\]
\[
\begin{pmatrix}
  b_{HH} & b_{HV} \\
  b_{VH} & b_{VV}
\end{pmatrix} = \text{diag}\{e^{-i\beta}, e^{i\beta}\},
\]
\[
\begin{pmatrix}
  l_{HH,2} & l_{HV,2} \\
  l_{VH,2} & l_{VV,2}
\end{pmatrix} = \mathbf{I},
\]
\[
\begin{pmatrix}
  l'_{HH,2} & l'_{HV,2} \\
  l'_{VH,2} & l'_{VV,2}
\end{pmatrix} = \mathbf{I},
\]

all of which are unimodular, as required.

S3: SUPPLEMENTAL FIGURES

FIG. S1. Hadamard walk (b) numerics and (c) experimental data of the intensity evolution for a 4-site circle (a) with \(|cc, H\rangle\) input complementing Figs 7 and 9 in the main text. Polarisation resolved averaged similarity over the first 12 steps \(S = 82.1\%\). The decrease of similarity is due to the high number of necessary EOM switches.